CARIBBEAN STEM OLYMPIADS

January 2024

Math Olympiad - Sample Problems and Solutions

All Levels

Note

Keep in mind that there may be many ways to solve a mathematics problem. The solutions presented here are just examples which will lead to the correct solution. Try to think of other ways to approach these problems!

Level I

Consumer Arithmetic

1. Julia purchased a new car at $\$18,000$. If the sale price is 25% less than the original price, what was the car's original price?

   Solution:
   Sale price $= (1 - 0.25)\cdot$ original price $= \$18,000$.
   $0.75p_o = 18000 \rightarrow p_o = 18000 \cdot \frac{1}{0.75} = 24000$.

2. The simple interest earned on a certain sum of money in 5 years at 9% per annum is 4016.25. Calculate the sum of money.

   Solution:

$$\begin{aligned} I &= P \cdot r \cdot t \\ 4016.25 &= P \cdot 0.09 \cdot 5 \\ P &= \frac{4016.25}{5 \cdot 0.09} = \$8925. \end{aligned}$$

3. John purchases a table on a hire purchase plan with a down payment of \$300 and 12 monthly payments of \$24 each month. The regular cash price of the same table is \$500. How much would John have saved if he purchased the table using cash?

   (a) \$212
   (b) \$56
   (c) <u>\$88</u>
   (d) \$288

   Solution:
   Savings = Cost of table on hire purchase - Cost of table in cash
   Savings = $\$(300 + 12(24)) - \$500 = \$588 - \$500 = \$88$.

Number Concepts

1. Water exists in the liquid state between $0°C$ and $100°C$. Given that the formula to convert degrees Celsius (C) to degrees Fahrenheit (F) is $F = (C \times \frac{9}{5}) + 32$, how many integer temperature values between $0°C$ and $100°C$ (excluding $0°C$ and $100°C$) can be converted to whole numbers in degrees Fahrenheit? Show/explain your reasoning for partial credit.

   Solution:
   Since $F = (C \times \frac{9}{5})$, the values of F are only whole when $C \times \frac{9}{5}$ is also whole. This happens when C is a multiple of 5. Therefore, this question simply boils down to finding the number of multiples of 5 between 0 and 100 exclusive. This number is 19.

2. A positive number n is 30% of five thirds of its cube. What is the value of n?

   (a) $\underline{\sqrt{2}}$
   (b) 2
   (c) $\sqrt[3]{2}$
   (d) $\frac{1}{\sqrt{2}}$

Solution:

$$
\begin{aligned}
n &= \frac{3}{10}\frac{5}{3}n^3 \\
n &= \frac{5}{10}n^3 \\
n^2 &= 2 \\
n &= \sqrt{2}
\end{aligned}
$$

3. Simplify the expression $\left(\frac{a^{\frac{1}{2}}\sqrt{a^3b^4\sqrt{c}}}{c^{-\frac{1}{4}}}\right)^{-1}$, expressing your answer in the form $\frac{1}{a^A b^B c^C}$ where A, B and C are positive numbers.

Solution:

$$
\begin{aligned}
\left(\frac{a^{\frac{1}{2}}\sqrt{a^3b^4\sqrt{c}}}{c^{-\frac{1}{4}}}\right)^{-1} &= \left(\frac{c^{-\frac{1}{4}}}{a^{\frac{1}{2}}\sqrt{a^3b^4\sqrt{c}}}\right) \\
&= \left(\frac{c^{-\frac{1}{4}}}{a^{\frac{1}{2}}a^{\frac{3}{2}}b^2c^{\frac{1}{4}}}\right) \\
&= \left(\frac{1}{a^{\frac{1}{2}}a^{\frac{3}{2}}b^2c^{\frac{1}{4}}c^{\frac{1}{4}}}\right) \\
&= \frac{1}{a^2b^2c^{\frac{1}{2}}}
\end{aligned}
$$

Fractions and Decimals

1. Reduce the following fraction to its lowest terms: $\frac{3}{4+\frac{2}{1-\frac{1}{4}}}$.

Solution:

$$
\frac{3}{4+\frac{2}{1-\frac{1}{4}}} = \frac{3}{4+\frac{2}{\frac{3}{4}}} = \frac{3}{4+\frac{2(4)}{3}} = \frac{3}{\frac{12+8}{3}} = \frac{9}{20}.
$$

2. Compute the following expression, giving your answer as a proper fraction.

$$\frac{1}{4} + 0.125 - 16^{-1}$$

   (a) $-15\frac{5}{8}$

   (b) $\frac{5}{16}$

   (c) $\frac{7}{16}$

   (d) $\frac{1}{12}$

   Solution:

$$\frac{1}{4} + \frac{1}{8} - \frac{1}{16} = \frac{4+2-1}{16} = \frac{5}{16}.$$

3. Evaluate

$$\frac{\frac{2}{5} + \frac{1}{6}}{\frac{12}{7} \times \frac{9}{4}}.$$

   Solution:

$$\frac{\frac{2}{5} + \frac{1}{6}}{\frac{12}{7} \times \frac{9}{4}} = \frac{\frac{2(6)+1(5)}{5(6)}}{\frac{12(9)}{7(4)}} = \frac{\frac{17}{30}}{\frac{108}{28}} = \frac{17 \times 28}{30 \times 108} = \frac{17 \times 7}{30 \times 27} = \frac{119}{810}.$$

Statistics and Probability

1. Leroy plays in a series of basketball matches. He scored 19 points on Monday, 23 points on Tuesday, 13 points on Wednesday, and 5 points on Thursday. On Friday, Leroy scored 3 points more than the mean (average) number of points he scored on the first four days. How many points did Leroy score in all?

   Solution:
   Total number of points on the first four days $= 19 + 23 + 13 + 5 = 60$
   Average number of points on first four days, $x_A = \frac{60}{4} = 15$
   Total number of points $= 60 + x_A + 3 = 60 + 15 + 3 = 78.$

2. A fair coin is flipped twice. What is the probability of getting at least one head?

Solution:

$$\begin{aligned}\mathbb{P}(\text{at least one head}) &= 1 - \mathbb{P}(\text{two tails in two tosses})\\ &= 1 - \mathbb{P}(\text{first toss tail}) \cdot \mathbb{P}(\text{second toss tail})\\ &= 1 - \left(\frac{1}{2}\right)^2\\ &= 1 - \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

Algebra

1. Evaluate the expression $\left(\frac{3a+2b}{2}\right)$ when $a = -3$ and $b = -4$.

Solution:

$$\frac{3(-3) + 2(-4)}{2} = \frac{-17}{2} = -8.5$$

2. Solve the following simultaneous equations for x and y.

$$\begin{aligned}2x + y &= 12\\ 6x + 5y &= 40\end{aligned}$$

Solution:

$$\begin{aligned}3 \times (2x + y &= 12)\\ -(6x + 5y &= 40)\end{aligned}$$

$$\begin{aligned}6x + 3y &= 36)\\ -(6x + 5y &= 40)\end{aligned}$$

This gives $-2y = -4 \rightarrow y = 2$.
Substituting this, we get $2x + 2 = 12 \rightarrow 2x = 10 \rightarrow x = 5$.

3. How many real solutions does $3x^2 - 2x + \frac{1}{3} = 0$ have?
   (a) 0
   (b) cannot be determined
   (c) <u>1</u>
   (d) 2

   Solution:
   First order of business is to calculate the discriminant, $b^2 - 4ac$.

   $$b^2 - 4ac = (-2)^2 - 4(3)(\frac{1}{3}) = 4 - 4 = 0.$$

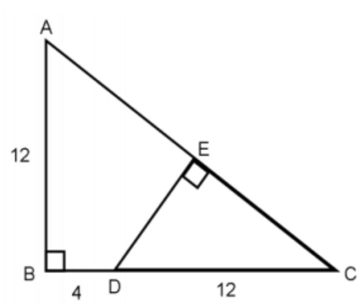
   Luckily for us, this discriminant equals 0, telling us this quadratic equation has exactly one real solution.

4. A car travels from Town A to Town B moving at a constant speed. The car starts at Town A and covers a distance of 1500 metres in 1 min 40 s. The car stops for 15 s and then resumes its journey. If the total journey took 3 min 30 s, what is the distance between the two towns?

   Solution:
   Speed of car $= \frac{1500\text{m}}{1\text{min}40s} = \frac{1500\text{m}}{100s} = 15\frac{\text{m}}{s}$.
   Total moving time of journey $= 3$ min $30s - 15s = 3$ min $15s = 195s$
   Total distance between the two towns $= 15\frac{\text{m}}{s} \cdot 195s = 15 \cdot (200 - 5) = 3000 - 75 = 2925\text{m}$

Geometry

1. In the diagram below,



(a) what is the length of the line AC?

(b) what is the product of the lengths DE and AC?

Solution:

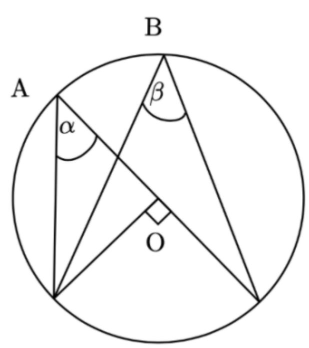
(a) By Pythagoras' theorem, $AC^2 = AB^2 + BC^2 = 12^2 + (12+4)^2 = 144 + 256 = 400$. $AC = \sqrt{400} = 20$.

(b) $\angle ACB = \angle DCE$ and $\angle ABC = \angle CED = 90°$, which gives that $\angle CDE = \angle BAC$. Therefore the triangles ABC and CDE are similar, with side DE corresponding to AB and DC corresponding to AC. Therefore, $DE = AB \times \frac{DC}{AC} = 12 \times \frac{12}{20} = \frac{144}{20}$. This gives $DE \times AC = \frac{144}{20} \times 20 = 144$.

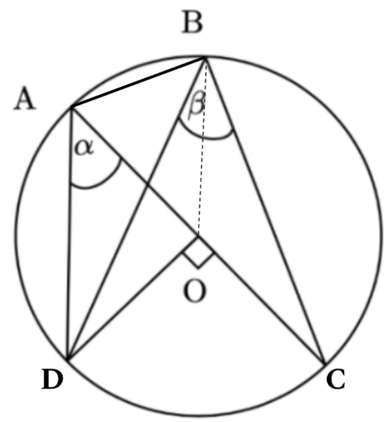
2. The length of the diagonal of a square is 1 metre. Compute the area of this square.

Solution:
Let the length of a side of the square be s. A diagonal drawn forms a right-angled isosceles triangle with two sides s. By Pythagoras' theorem, the length of the diagonal (hypotenuse) $= \sqrt{s^2 + s^2} = \sqrt{2}s$. If the length of the diagonal is 1, then $\sqrt{2}s = 1 \rightarrow s = \frac{1}{\sqrt{2}}$. The area of a square with sides s is s^2, therefore the area of the square is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

3. You are given the circle below. The angle β is bisected by a line OB (not shown). If the diameter of the circle is D, what is the length of the arc AB?



Solution:
According to the Central Angle Theorem, the central angle subtended by two points on the circumference of a circle is twice the inscribed angle subtended by those same points. This means for our circle as drawn, the angle β equals $\frac{90°}{2} = 45°$.



We also use the fact that inscribed angles in a semicircle are 90°. Therefore, $\angle ABC = 90°$. Since $\beta = 45°$, then $\angle ABD = 90 - 45 = 45°$.
With the bisector OB drawn, we know that $\angle OBA = 45 + 22.5 = 67.5°$.
As both OB and OA are both radii of equal length, let's say, r, the $\triangle AOB$ is isosceles with equal angles $\angle OAB = \angle OBA = 67.5°$.
This gives the $\angle AOB = 180 - 2(67.5) = 180 - 135 = 45°$.

Now that we have this information, we can find the length of the arc AB.

$$\text{Arc length} = \pi \cdot D \cdot \frac{45}{360} = \frac{\pi D}{8}$$

Level II

Algebra

1. We are given that $x^4 - y^4 = 13$ and $x^2 + y^2 = 4$. What is the value of $(x+y)^2$?

   Solution:
   By the difference of two squares, $(a^2 - b^2) = (a+b)(a-b)$. Therefore, $x^4 - y^4 = (x^2)^2 + (y^2)^2 = (x^2 + y^2)(x^2 - y^2)$. From this, we get that $(x^2 - y^2) = \frac{x^4-y^4}{x^2+y^2} = \frac{13}{4}$. We now have

$$x^2 + y^2 = 4 \tag{1}$$
$$x^2 - y^2 = \frac{13}{4} \tag{2}$$

   Adding (1) and (2), we get that $2x^2 = \frac{29}{4}$. By (1) - (2), we get that $2y^2 = \frac{3}{4}$. Thus, $4x^2y^2 = \frac{29*3}{4*4} = \frac{87}{16}$. $2xy = \pm\sqrt{\frac{87}{16}}$.

$$(x+y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2xy = 4 \pm \frac{\sqrt{87}}{4}.$$

2. For what positive real values of k will the equation $kx^2 - (k+1)x - 1 = 0$ have exactly one real solution?

   Solution:
   First order of business is to calculate the discriminant, $b^2 - 4ac$.

   $b^2 - 4ac = (-(k+1))^2 - 4(k)(-1) = (k+1)^2 + 4 = 0 \to (k+1)^2 = -4.$

   This has no real-valued solutions.

3. Which of the following values of x satisfies the inequality $x^2 + 8x + 15 < 0$?

   (a) $\underline{x = -4}$

(b) $x = -2$

(c) $x = -5$

(d) $x = -3$

Solution:

$$\begin{aligned} x^2 + 8x + 15 &= 0 \\ (x+3)(x+5) &= 0 \\ x+3 &= 0,\ x+5 = 0 \\ x &= -3,\ x = -5 \end{aligned}$$

In order to satisfy the inequality, values must either be $x < -3$, $x \neq -5$ and $x < -5$. The only provided value which satisfies these conditions is $x = -4$.

4. Find the two values of x for which $x^{1+\log_{10} x} = 10^4 \cdot x$.

Solution:

$$\begin{aligned} x^{1+\log_{10} x} &= 10^4 \cdot x \\ x \cdot x^{\log x} &= 10^4 \cdot x \\ x^{\log x} &= 10^4 \\ \text{Let } x &= 10^y. \\ (10^y)^{\log 10^y} &= 10^4 \\ (10^y)^y &= 10^4 \\ 10^{y^2} &= 10^4 \\ y^2 &= 4 \\ y &= \pm 2 \rightarrow x = 10^2, 10^{-2} \end{aligned}$$

Geometry

1. Find the equation of the sphere whose diameter has endpoints $(-2, 3, 1)$ and $(0, 5, -1)$.

Solution:

Centre of the sphere = Midpoint of the diameter, $M = \left(\frac{-2+0}{2}, \frac{3+5}{2}, \frac{1-1}{2}\right) = ($

Length of diameter, $l_D = \sqrt{(-2)^2 + (3-5)^2 + (1-(-1)^2} = \sqrt{12} = 2\sqrt{3}$.

$$\text{Length of radius, } l_r = \sqrt{3}.$$

This gives the equation $(x+1)^2 + (y-4)^2 + z^2 = (\sqrt{3})^2 = (x+1)^2 + (y-4)^2 + z^2 = 3$.

2. A solid wooden cube is painted red on the outside. The cube is then cut into 27 smaller identical cubes. What fraction of all of the smaller cubes' sides are painted red?

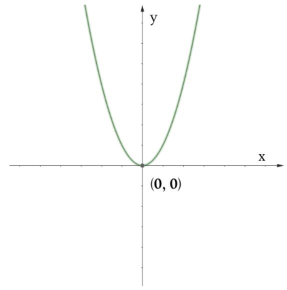
   (a) $\frac{1}{6}$
   (b) $\frac{2}{7}$
   (c) $\underline{\frac{1}{3}}$
   (d) $\frac{1}{2}$

   Solution:
   A cube has 6 faces. In order to cut one larger cube into 27 identical ones, one would need to make 4 cuts (2 top-down and 2 left-right), resulting in the formation of 9 faces for each face of the original cube. This gives $9 \cdot 6 = 54$ painted smaller cube faces. As there are 27 cubes, and a cube has 6 faces, there are a total of 162 faces. The fraction of the total number of sides painted red is $\frac{54}{162} = \frac{1}{3}$.

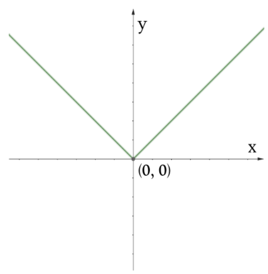
3. Rapid Plot! Sketch the following in 1 minute (note that a is a positive integer):
   $y = ax^2$, $y = |x|$, $y = e^{-ax}$ and $y = \log(\frac{x}{a})$.

Solution:

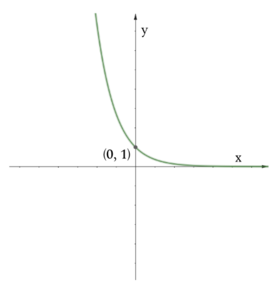
1. $y = ax^2$



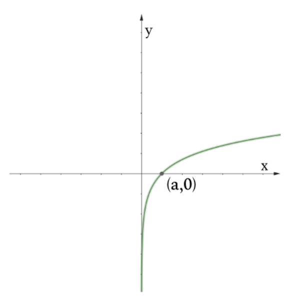
2. $y = |x|$



3. $y = e^{-ax}$



4. $y = \log(\frac{x}{a})$



Trigonometry

1. If $\cos A = \frac{3}{5}$, find $\sin 2A$.
   Solution:
   $$\sin 2A = \sin(A + A) = 2 \sin A \cos A.$$
   Also, recall that $\sin^2 A + \cos^2 A = 1$. Therefore, if $\cos A = \frac{3}{5}$, then $\sin A = \sqrt{1 - (\frac{3}{5})^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Then, $\sin 2A = 2(\frac{3}{5})(\frac{4}{5}) = \frac{24}{25}$.

2. If $\tan(x + y) = \sqrt{3}$ and $\tan(x - y) = \frac{1}{\sqrt{3}}$, where $(x + y) \in (0°, 90°)$ and $x > y$, find x and y.

   Solution:
   $$\begin{aligned} \tan(x+y) &= \sqrt{3} \\ \tan(x+y) &= \tan 60° \\ x + y &= 60 \end{aligned}$$
   $$\begin{aligned} \tan(x-y) &= \frac{1}{\sqrt{3}} \\ \tan(x-y) &= \tan 30° \\ x - y &= 30 \end{aligned}$$
   Solving the simultaneous equations, we gather that $x + y + x - y = 90 \rightarrow 2x = 90 \rightarrow x = 45$. This gives $y = 45 - 30 = 15$.

3. Given that $\sec \theta = \frac{7}{3}$, find the value of $\sin 2\theta$.
   (a) $\underline{\frac{12\sqrt{10}}{49}}$
   (b) $\frac{4\sqrt{10}}{3}$
   (c) $\frac{3}{7}$
   (d) $\frac{2\sqrt{10}}{7}$

Solution:
Since $\sec\theta = \frac{1}{\cos\theta}$, $\cos\theta = \frac{3}{7}$. It follows from the definition of cosine that a right-angled triangle with the angle θ as described has a hypotenuse of 7 units and adjacent side length of 3 units. By Pythagoras' theorem, the length of the opposite side x can be found: $x^2 = 7^2 - 3^2 = 40 \rightarrow x = \sqrt{40} = 2\sqrt{10}$ units.

We deduce from this that $\sin\theta = \frac{2\sqrt{10}}{7}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\Big(\frac{2\sqrt{10}}{7}\Big)\Big(\frac{3}{7}\Big) = \frac{12\sqrt{10}}{49}.$$

Vectors

1. Consider a row vector P, consisting of 6 elements, and a column vector Q, also consisting of 6 elements. In which order can vector multiplication be performed?

   (a) PQ only
   (b) QP only
   (c) <u>Both ways</u>
   (d) Neither

   Solution:
   Both ways. P is a 1×6 vector, and Q is a 6×1 vector. PQ yields a scalar, while QP yields a square matrix.

2. If $a = 2i + k$ and $b = -3j + 2k$ (where i is the unit vector along the x axis, j is the unit vector along the y axis, and k is the unit vector along the z axis), find the unit vector u that is opposite in direction to $v = 2a - b$.

Solution:

$$a = 2i + k$$

$$b = -3j + 2k$$

$$v = 2a - b$$

$$2a = 2(2i + k) = 4i + 2k$$

$$2a - b = 4i + 2k - (-3j + 2k) = 4i + 3j$$

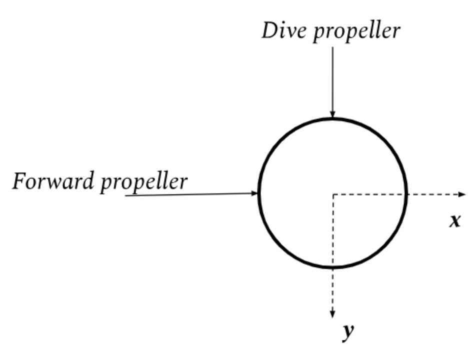
$$v = 4i + 3j, |v| = \sqrt{4^2 + 3^2 + 0^2} = 5$$

By the definition of the unit vector,

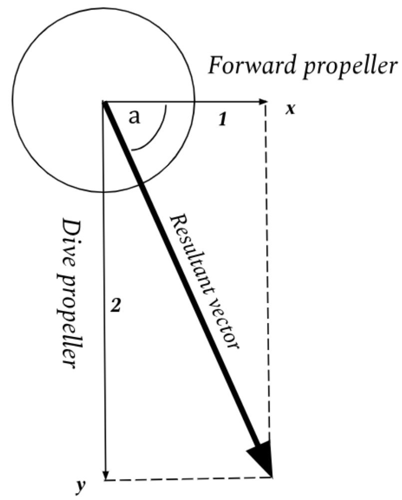
$$u = -\frac{v}{|v|}$$

$$u = -\frac{4}{5}i - \frac{3}{5}j$$

3. A spherical underwater robot has 2 propellers: one to dive and one to move forward. The dive propeller applies a net force (beyond buoyancy) that is twice as strong as the forward propeller. Complete the diagram to show the direction of the robot's movement.



Solution:



Resultant vector magnitude: $\sqrt{1^2+2^2} = \sqrt{5}$.
Resultant vector angle below horizontal axis: $a = \tan^{-1}(2)$

Exponentials and Logarithms

1. The decibel (dB) is a unit of measurement for the loudness of sounds. A $10-$fold increase in loudness corresponds to an increase of $10dB$ ($100-$fold is $+20dB$, $1000-$fold is $+30dB$, and so on).

   (a) Write the relation between the fold-loudness of a sound and its decibel value. You may use S_0 and S_1 to represent the sound intensities.

   (b) You're sitting at a music mixing board and are told to make the speaker output twice as loud. How many decibels should you increase the output level of the speaker by?

   Solution:

   (a) It's necessary to recognize that the scale is logarithmic. From there, we can write an equation to determine the desired relationship. Let

$\frac{S_1}{S_0} = 100$ and given the corresponding decibel increase of +20:

$$\begin{aligned} 20 &= C \log_{10} 100dB \\ C &= 2C \\ C &= 10 \end{aligned}$$

Therefore, the relation is $10 \log_{10} \frac{S_1}{S_0}$ dB.

(b) Increase decibel output by $10 \log_{10} 2$.

$$10 \log_{10} 2x = 10(\log_{10} x + \log_{10} 2) = 10 \log_{10} x + 10 \log_{10} 2.$$

2. If $xy^m = yx^3$, then solve for m.

Solution:

$$\begin{aligned} xy^m &= yx^3 \\ y^{m-1} &= x^2 \\ (m-1)\log y &= 2 \log x \\ m-1 &= 2\frac{\log x}{\log y} \\ m &= 1 + 2\frac{\log x}{\log y} \end{aligned}$$

Recognizing this as change of base, we get

$$m = 1 + 2 \log_x y$$

3. Suppose that the population of a colony of bacteria increases exponentially. If the population at the start is 300 and 4 hours later it is 1800, how long will it take for the population to reach 3000?

(a) $10\frac{\ln 4}{\ln 6}$

(b) $\underline{4\frac{\ln 10}{\ln 6}}$

(c) $4\frac{\ln 6}{\ln 10}$

(d) $6\frac{\ln 10}{\ln 4}$

Solution:
Let the population of the colony be $P(t)$.

$$P(t) = 300e^{kt}$$
$$1800 = 300e^{4k} \to 6 = e^{4k} = k = \frac{\ln 6}{4}$$
$$3000 = 300e^{\frac{\ln 6}{4}t} \to t = \frac{4\ln 10}{\ln 6}$$

4. Find the positive integer solution of $\log_3 x \log_4 x \log_6 x = \log_3 x \log_4 x + \log_4 x \log_6 x + \log_3 x \log_6 x$.

Solution:
Divide both sides through by $\log_3 x \log_4 x \log_6 x$. This yields

$$1 = \frac{1}{\log_6 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x}$$

By change of base, we then get

$$\begin{aligned} 1 &= \frac{1}{\frac{\log x}{\log 6}} + \frac{1}{\frac{\log x}{\log 3}} + \frac{1}{\frac{\log x}{\log 4}} \\ &= \frac{\log 6}{\log x} + \frac{\log 3}{\log x} + \frac{\log 4}{\log x} \\ \log x &= \log 6 + \log 3 + \log 4 \\ &= \log (6 * 3 * 4) = \log 72 \end{aligned}$$

Therefore, $x = 72$.

Calculus

1. A girl sits on a cliff 20 meters above the ground. She throws a tennis ball vertically upwards from the cliff. After t seconds it has height s above the ground which follows:

$$s^2 = 20 + 50t - t^2$$

What is the maximum height of the ball?

Solution:
To find the maximum height of the ball, we first need to find the value of t for which the value of s is maximized. We do this by differentiating the equation given, equating it to 0, and solving for t.

$$\begin{aligned}
\frac{ds}{dt} &= 50 - 2t = 0 \\
2t &= 50 = 25 \\
s(t = 25) &= 20 + 50(25) - (25)^2 \\
s_{max} &= 20 + (25)(25) \\
&= 645
\end{aligned}$$

2. Find the function, $y(x)$, that satisfies the differential equation:

$$\left(\frac{dy}{dx}\right)^2 = \frac{1 - y^2}{1 - x^2}$$

Solution:

$$\begin{aligned}
\left(\frac{dy}{dx}\right)^2 &= \frac{1 - y^2}{1 - x^2} \\
\frac{dy}{dx} &= \sqrt{\frac{1 - y^2}{1 - x^2}} \\
\sqrt{\frac{1}{1 - y^2}}dy &= \sqrt{\frac{1}{1 - x^2}}dx \\
\int \sqrt{\frac{1}{1 - y^2}}dy &= \int \sqrt{\frac{1}{1 - x^2}}dx \\
\arcsin y + C_1 &= \arcsin x + C_2 \\
\arcsin y &= \arcsin x + C \\
y &= \sin(\arcsin x + C)
\end{aligned}$$

3. Find the second derivative of $\sin(x^2-4)$ with respect to x.

   (a) $4x^2\sin(x^2-4) - 2\cos(x^2-4)$

   (b) $\underline{-4x^2\sin(x^2-4) + 2\cos(x^2-4)}$

   (c) $2x^2\sin(x^2-4) + 2\cos(x^2-4)$

   (d) $2x^2\sin(x^2-4) + 4\cos(x^2-4)$

   Solution:
   With chain rule, first derivative: $\frac{d}{dx}(\sin(x^2-4)) = \frac{d}{dx}(x^2-4)\cos(x^2-4) = 2x\cos(x^2-4)$.
   With product and chain rules, second derivative: $\frac{d}{dx}(2x)\cos(x^2-4) + 2x\frac{d}{dx}(\cos(x^2-4))$
   $= 2\cos(x^2-4) + 2x(2x)(-\sin(x^2-4)) = -4x^2\sin(x^2-4) + 2\cos(x^2-4)$.

Level III

Series and Sequences

1. What is the sum of the infinite series
$$\frac{7}{4}-\frac{7}{4^2}+\frac{7}{4^3}-\cdots?$$
Solution:
The general form of an infinite geometric series is $a_1+a_1r+a_1r^2+a_1r^3+\cdots$.
$$r=\frac{-\frac{7}{4^2}}{\frac{7}{4}}=\frac{\frac{7}{4^3}}{-\frac{7}{4^2}}=-\frac{1}{4}.$$
Since $|r|<1$, the sum of this series is $S=\frac{a_1}{1-r}$.
$$S=\frac{\frac{7}{4}}{1-\left(-\frac{1}{4}\right)}=\frac{\frac{7}{4}}{\frac{5}{4}}=\frac{7}{5}.$$

2. The sequence $x_n=\frac{1}{n}, n\in\mathbb{Z}^+$ is:
   (a) <u>Bounded and Convergent</u>
   (b) Bounded and Divergent
   (c) Unbounded and Convergent
   (d) Unbounded and Divergent

   Solution:
   - A sequence is bounded if there exists two numbers such that all the terms of the sequence lie within the range produced by these two numbers. In other words, the sequence must be bounded above and below.

     The largest value in this sequence is 1, and there is no value in this sequence that is less than 0. Therefore, any number greater than 1 serves as an upper bound for this sequence. Any number less than 0 serves as a lower bound for this sequence. Since this is bounded above and below, the sequence is **bounded**.

- A sequence is convergent if the value of the terms of the sequence tend to a particular value.

  As $n \to \infty$, $\frac{1}{n} \to 0$. Therefore, this sequence is **convergent**.

Euclidean Geometry

1. Consider a circle with three distinct points labelled A, B and C on its circumference. What is the minimum number of straight lines/line segments which must be drawn to locate the centre of this circle?

   (a) 6 lines

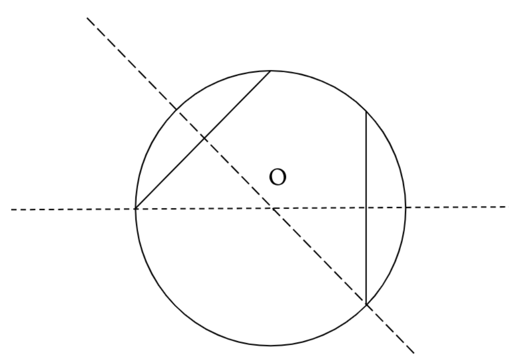
   (b) 2 lines

   (c) 3 lines

   (d) <u>4 lines</u>

   Solution:
   The correct answer is 4 lines. You can find the center by following this procedure:

   - Start by drawing two chords on the circle (shown as solid lines here).
   - Then draw perpendicular bisectors of each of the original chords.
   - Where the two bisectors intersect is the center of the circle.

2. Let A be a point in $\mathbb{R}^3$ with Cartesian coordinates (x_0, y_0, z_0). A can be expressed in spherical coordinates as $(\rho_0, \theta_0, \phi_0)$. The following equations can be used to convert from spherical coordinates to Cartesian

coordinates:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \text{for } \rho \geq 0, \theta \in [0, 2\pi) \text{ and } \phi \in [0, \pi).$$

Find $\phi_0(x_0, y_0, z_0)$.

Solution:

$$\begin{aligned} x_0^2 + y_0^2 &= \rho_0^2 \sin^2\phi_0 \cos^2\theta_0 + \rho_0^2 \sin^2\phi_0 \sin^2\theta_0 \\ &= \rho_0^2 \sin^2\phi_0 (\cos^2\theta_0 + \sin^2\theta_0) \\ &= \rho_0^2 \sin^2\phi_0 \\ \sqrt{x_0^2 + y_0^2} &= \rho_0 \sin\phi_0 \end{aligned}$$

$$z_0 = \rho_0 \cos\phi_0$$

Combining, we get

$$\begin{aligned} \frac{\sqrt{x_0^2 + y_0^2}}{z_0} &= \frac{\rho_0 \sin\phi_0}{\rho_0 \cos\phi_0} \\ &= \tan\phi_0 \\ \tan\phi_0 &= \frac{\sqrt{x_0^2 + y_0^2}}{z_0} \\ \phi_0 &= \arctan\left(\frac{\sqrt{x_0^2 + y_0^2}}{z_0}\right). \end{aligned}$$

Calculus

1. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin\theta} d\theta$.

   Solution:
   Let $u = 1+\sin\theta \to du = \cos\theta d\theta$. When $\theta = \frac{\pi}{2}, u = 1+\sin\frac{\pi}{2} = 2$ and $\theta = 0, u = 1+\sin 0 = 1$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin\theta} d\theta &= \int_1^2 \frac{du}{u} \\ &= \ln|u|\Big|_1^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2.\end{aligned}$$

2. Integrate $\int x\cos x dx$.

   (a) $x\cos x - \sin x$

   (b) $x\sin x - \cos x$

   (c) $\underline{x\sin x + \cos x}$

   (d) $x\cos x + \sin x$

   Solution:
   By the product rule of integration,

$$\begin{aligned}\int x\cos x dx &= x\int \cos x dx - \int \left[\frac{d}{dx}(x)\int \cos x dx\right] dx \\ &= x\sin x + C_1 - \int \sin x dx \\ &= x\sin x + C_1 + \cos x + C_2 \\ &= x\sin x + \cos x + C\end{aligned}$$

3. Simplify $\int \sin^3(t)\cos^2(t)dt$.

Solution:

$$\int \sin^3(t)\cos^2(t)dt = \int \sin^2(t)\cos^2(t)\sin(t)dt$$

Recall that $1-\cos^2(t) = \sin^2(t)$. Substituting this, we get

$$\int \sin^3(t)\cos^2(t)dt = \int (1-\cos^2(t))\cos^2(t)\sin(t)dt$$

Let $u = \cos t$. Then, $du = -\sin t dt \to \sin t dt = -du$.

$$\begin{aligned}\int \sin^3(t)\cos^2(t)dt &= -\int (1-u^2)u^2du \\ &= -\int u^2 - u^4 du \\ &= -\Big[\frac{u^3}{3} - \frac{u^5}{5} + C\Big] \\ &= -\frac{\cos^3(t)}{3} + \frac{\cos^5(t)}{5} + C.\end{aligned}$$

Vectors and Matrices

1. Find the determinant of the matrix $\begin{bmatrix} 1 & -4 & 7 & 5 \\ 0 & 3 & 1 & -6 \\ 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 1 & -4 & 7 & 5 \\ 0 & 3 & 1 & -6 \\ 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 & -6 \\ -1 & 7 & 0 \\ 0 & 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 & -6 \\ 1 & 7 & 0 \\ 0 & 0 & 2 \end{vmatrix} + 7\begin{vmatrix} 0 & 3 & -6 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 & 1 \\ 1 & -1 & 7 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -4 & 7 & 5 \\ 0 & 3 & 1 & -6 \\ 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1\Big(3\begin{vmatrix} 7 & 0 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} -1 & 7 \\ 0 & 0 \end{vmatrix}\Big)$$

$$+ 4\Big(0\begin{vmatrix} 7 & 0 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}\Big)$$

$$+ 7\Big(0\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\Big)$$

$$- 5\Big(0\begin{vmatrix} -1 & 7 \\ 0 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\Big)$$

$$= (3(14) - 1(-2) + 0) + 4(0 - 1(2) + 0) + 7(0 - 3(2) + 0) - 5(0 + 0$$

$$= 42 + 2 - 8 - 42 = -6.$$

2. The eigenvalues, λ, of an $n \times n$ matrix, A, can be found by solving the equation $det(A - \lambda I) = 0$ where I is the $n \times n$ identity matrix. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$.

   Solution:

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (2-\lambda)(3-\lambda) - 1(-1) = 0$$

$$6 - 5\lambda + 1 + \lambda^2 = 0 \to \lambda^2 - 5\lambda + 7 = 0$$

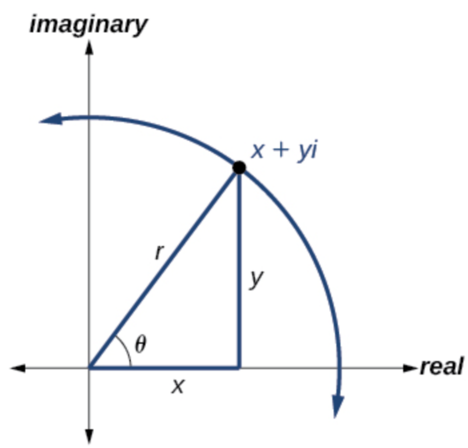
   Solving this quadratic equation, we get that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{5^2 - 4(7)}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5 \pm i\sqrt{3}}{2}.$$

Complex Numbers

1. The complex number, z, can be represented by a vector in the complex plane. What is the effect on the vector representing z in the complex plane, of multiplying z by $re^{i\theta}, r \in (0, \infty), \theta \in [-\pi, \pi)$?

   Solution:
   Multiplying z by $re^{i\theta}$ will make the vector r times as long and rotate it an angle of θ counterclockwise around the origin.
   Consider a complex number in polar form: $z = r_1 e^{i\theta_1}$. This means that the complex number when represented on the complex plane is a line with magnitude (length r) positioned at an angle of θ above the positive real axis as shown.

   With this knowledge, we can see that if we multiple the complex number z by $re^{i\theta}$, we get
   $$r_1 e^{i\theta_1} \cdot re^{i\theta} = r_1 r e^{i(\theta+\theta_1)}$$
   which represents a new complex number of magnitude rr_1 and argument $\theta + \theta_1$ above the positive real axis, as our answer describes.

2. Find the purely real solution to $4z^2 + (8 + 4i)z + (3 + 2i) = 0$.

   Solution:
   This equation can be solved directly using the quadratic formula of

course, but this is a rather tedious approach. You can more quickly solve this by recognizing that only the real solution of the equation is desired. Therefore, let $z = a$, where $a \in \mathbb{R}$. The equation becomes:

$$4a^2 + (8+4i)a + (3+2i) = 0$$

$$4a^2 + 8a + 3 + i(4a+2) = 0$$

For this equation to be satisfied, both the real and imaginary part of the complex number must equal 0. For this to be the case, $4a + 2 = 0$ and $4a^2 + 8a + 3 = 0$. The only value that satisfies both equations is $z = -\frac{1}{2}$.

Differential Equations

1. Find the general solution for the differential equation $\frac{dy}{dx} = x + xy^2$.

   Solution:
   We proceed by separation of valuables.

$$\begin{aligned} \frac{dy}{dx} &= x + xy^2 \\ dy &= x(1+y^2)dx \\ \frac{1}{1+y^2}dy &= xdx \\ \int \frac{1}{1+y^2}dy &= \int xdx \\ \arctan y &= \frac{x^2}{2} + C \\ y &= \tan\left(\frac{x^2}{2} + C\right) \end{aligned}$$

2. Solve the differential equation $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, where $x(0) = 0$ and $\frac{dx}{dt}(0) = 5$.

   (a) $5e^{-t} - 5e^{6t}$

   (b) $5e^{-t} + 5e^{6t}$

(c) $5e^{-2t} + 5e^{-3t}$

(d) $\underline{5e^{-2t} - 5e^{-3t}}$

Solution:
Let $x_h(t)$ be of the form $x_h(t) = Ae^{\lambda t}$ where both A and λ can be complex. Then,

$$\lambda^2 A + 5A\lambda + 6A = 0$$

Solving for λ, we get

$$\begin{aligned} \lambda^2 A + 5A\lambda + 6A &= 0 \\ (\lambda_1 + 2)(\lambda_2 + 3)A &= 0 \\ \lambda_1 = -2, \lambda_2 &= -3. \end{aligned}$$

Hence, $x(t) = A[e^{-2t} + e^{-3t}] = C_1 e^{-2t} + C_2 e^{-3t}$.

When $x(0) = 0$, $0 = C_1 + C_2 = 0 \rightarrow C_1 = -C_2$.

We also know that $\frac{dx}{dt}(0) = 5$, therefore

$$\begin{aligned} -2C_1 e^{-2(0)} - 3C_2 e^{-3(0)} &= 5 \\ -2C_1 - 3C_2 &= 5 \\ -2C_1 + 3C_1 &= 5 \\ C_1 &= 5 \\ C_2 = -C_1 &= -5 \end{aligned}$$

Therefore, solution is $5e^{-2t} - 5e^{-3t}$.