# CSO Math Olympiad - Sample Problems

Sample Questions All Levels

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# 1 Level I

## 1.1 Consumer Arithmetic

1. Julia purchased a new car at $\$18,000$. If the sale price is 25% less than the original price, what was the car's original price?

   **Solution:** $\$24,000$.

   Sale price $= (1-0.25)\cdot$ original price $= \$18,000$.
   $0.75p_o = 18000 \to p_o = 18000 \cdot \frac{1}{0.75} = 24000$.

2. The simple interest earned on a certain sum of money in 5 years at 9% per annum is 4016.25. Calculate the sum of money.

   **Solution:** $8925.

$$\begin{aligned} I &= P \cdot r \cdot t \\ 4016.25 &= P \cdot 0.09 \cdot 5 \\ P &= \frac{4016.25}{5 \cdot 0.09} = 8925. \end{aligned}$$

3. John purchases a table on a hire purchase plan with a down payment of $300 and 12 monthly payments of $24 each month. The regular cash price of the same table is $500. How much would John have saved if he purchased the table using cash?

   (a) $212
   (b) $56
   (c) $88
   (d) $288

   **Solution:**

   Savings = Cost of table on hire purchase - Cost of table in cash
   Savings $= \$(300 + 12(24)) - \$500 = \$588 - \$500 = \$88$.

4. A business owner budgets $\$20,000$ to purchase furniture for the office. The owner must purchase a minimum of 100 items (in total) to qualify for a discount. If the owner pays $150 per chair and $250 per desk, what is the **maximum number of desks** the owner can purchase while staying within the budget and meeting the **minimum item requirement**?

   **Solution:** 50 desks.

Let $c$ represent the number of chairs and $d$ represent the number of desks bought. We can then represent the constraints as follows:

$$c + d \geq 100 \quad (1)$$
$$150c + 250d \leq 20000 \quad (2)$$

The objective is to maximize $d$ subject to constraints (1) & (2). Graphing (or sketching) the lines $c + d = 100$ and $150c + 250d = 20000$ and identifying the feasible region (from the constraints) will reveal that the maximum feasible number of desks occurs at the intersection of the 2 lines. Therefore, we can solve for $d$ at this point as follows:

$$\begin{aligned} c &= 100 - d && (1) \\ 150(100 - d) + 250d &= 20000 && (2) \\ \implies 100d &= 5000 \\ \implies d &= 50 \end{aligned}$$

Therefore, the maximum number of desks that will not violate either of the constraints is 50.

5. Hannah is shopping at a book store that is running a buy 2 get 1 free promotion. If each book costs \$35 and she leaves the store with 7 books, how much did she pay?

   **Solution:** \$175

   "Buy 2 get 1 free" effectively means "get 3 books for the price of 2". $7 = 2(3) + 1$, so Hannah got 2 sets of 3 books plus 1 single book. Therefore, she would have paid $2(2 \cdot \$35) + 1(\$35) = \$175$.

## 1.2 Number Concepts

1. Water exists in the liquid state between $0°C$ and $100°C$. Given that the formula to convert degrees Celsius ($C$) to degrees Fahrenheit ($F$) is $F = (C \times \frac{9}{5}) + 32$, how many integer temperature values between $0°C$ and $100°C$ (excluding $0°C$ and $100°C$) can be converted to whole numbers in degrees Fahrenheit? Show/explain your reasoning for partial credit.

   **Solution:** 19.

   Since $F = (C \times \frac{9}{5})$, the values of $F$ are only whole when $C \times \frac{9}{5}$ is also whole. This happens when $C$ is a multiple of 5. Therefore, this question simply boils down to finding the number of multiples of 5 between 0 and 100 exclusive. This number is 19.

2. A positive number $n$ is 30% of five thirds of its cube. What is the value of $n$?

   (a) $\sqrt{2}$

   (b) 2

   (c) $\sqrt[3]{2}$

(d) $\frac{1}{\sqrt{2}}$

**Solution:**

$$\begin{aligned} n &= \frac{3}{10}\frac{5}{3}n^3 \\ n &= \frac{5}{10}n^3 \\ n^2 &= 2 \\ n &= \sqrt{2} \end{aligned}$$

3. Simplify the expression $\left(\frac{a^{\frac{1}{2}}\sqrt{a^3b^4\sqrt{c}}}{c^{-\frac{1}{4}}}\right)^{-1}$, expressing your answer in the form $\frac{1}{a^A b^B c^C}$ where $A, B$ and $C$ are positive numbers.

   **Solution:** $\frac{1}{a^2b^2c^{\frac{1}{2}}}$.

$$\begin{aligned} \left(\frac{a^{\frac{1}{2}}\sqrt{a^3b^4\sqrt{c}}}{c^{-\frac{1}{4}}}\right)^{-1} &= \left(\frac{c^{-\frac{1}{4}}}{a^{\frac{1}{2}}\sqrt{a^3b^4\sqrt{c}}}\right) \\ &= \left(\frac{c^{-\frac{1}{4}}}{a^{\frac{1}{2}}a^{\frac{3}{2}}b^2c^{\frac{1}{4}}}\right) \\ &= \left(\frac{1}{a^{\frac{1}{2}}a^{\frac{3}{2}}b^2c^{\frac{1}{4}}c^{\frac{1}{4}}}\right) \\ &= \frac{1}{a^2b^2c^{\frac{1}{2}}} \end{aligned}$$

4. Compute the sum of all the numbers beginning with 1 and ending with 50. **(CW)**

   **Answer:** 1275.

5. What is the simplest form of the expression $\frac{4^{x+1}}{16^{x-2}}$ given that $x = 3$

   (a) $2^2$

   (b) $4^6$

   (c) $\underline{4^2}$

   (d) $2^3$

   **Answer:** $4^2$.

6. Simplify this fraction so that there are no roots in the denominator: $\frac{5}{\sqrt{3}+2}$

   **Answer:** $10 - 5\sqrt{3}$.

## 1.3 Fractions and Decimals

1. Reduce the following fraction to its lowest terms: $\frac{3}{4+\frac{2}{1-\frac{1}{4}}}$.

   **Solution:** $\frac{9}{20}$.

$$\frac{3}{4+\frac{2}{1-\frac{1}{4}}} = \frac{3}{4+\frac{2}{\frac{3}{4}}} = \frac{3}{4+\frac{2(4)}{3}} = \frac{3}{\frac{12+8}{3}} = \frac{9}{20}.$$

2. Compute the following expression, giving your answer as a proper fraction.

$$\frac{1}{4} + 0.125 - 16^{-1}$$

   (a) $-15\frac{5}{8}$
   (b) $\frac{5}{16}$
   (c) $\frac{7}{16}$
   (d) $\frac{1}{12}$

   **Solution:** $\frac{5}{16}$.

$$\frac{1}{4} + \frac{1}{8} - \frac{1}{16} = \frac{4+2-1}{16} = \frac{5}{16}.$$

3. Evaluate

$$\frac{\frac{2}{5}+\frac{1}{6}}{\frac{12}{7}\times\frac{9}{4}}.$$

   **Solution:** $\frac{119}{810}$.

$$\frac{\frac{2}{5}+\frac{1}{6}}{\frac{12}{7}\times\frac{9}{4}} = \frac{\frac{2(6)+1(5)}{5(6)}}{\frac{12(9)}{7(4)}} = \frac{\frac{17}{30}}{\frac{108}{28}} = \frac{17\times 28}{30\times 108} = \frac{17\times 7}{30\times 27} = \frac{119}{810}.$$

4. A bakery makes 240 cookies ($\frac{2}{3}$ are chocolate,$\frac{1}{4}$ are oatmeal, the rest almond) If the bakery sold $\frac{5}{8}$ of the chocolate, $\frac{3}{10}$of the oatmeal, and $\frac{1}{2}$ of the almond, **what fraction of the total cookies are unsold?**

   **Answer:** $\frac{7}{15}$.

5. A book has 180 pages. $\frac{1}{5}$ are dedicated to the introduction, $\frac{2}{9}$ are dedicated to the first chapter, and the remaining pages are dedicated to the second chapter. How many pages are dedicated to the second chapter?

   **Answer:** 104.

6. Simplify: $\frac{1840}{120}$

(a) $15\frac{1}{3}$
(b) $\frac{47}{3}$
(c) $\frac{92}{6}$
(d) $\frac{92}{5}$

**Answer:** $15\frac{1}{3}$.

7. Convert 0.015625 to a fraction in its lowest terms.

**Answer:** $\frac{1}{64}$.

## 1.4 Statistics and Probability

1. Leroy plays in a series of basketball matches. He scored 19 points on Monday, 23 points on Tuesday, 13 points on Wednesday, and 5 points on Thursday. On Friday, Leroy scored 3 points more than the mean (average) number of points he scored on the first four days. How many points did Leroy score in all?

**Solution:** 78 points.

Total number of points on the first four days $= 19 + 23 + 13 + 5 = 60$
Average number of points on first four days, $x_A = \frac{60}{4} = 15$
Total number of points $= 60 + x_A + 3 = 60 + 15 + 3 = 78$.

2. A fair coin is flipped twice. What is the probability of getting at least one head?

**Solution:** $\frac{3}{4}$.

$$\begin{aligned}\mathbb{P}(\text{at least one head}) &= 1 - \mathbb{P}(\text{two tails in two tosses})\\ &= 1 - \mathbb{P}(\text{first toss tail}) \cdot \mathbb{P}(\text{second toss tail})\\ &= 1 - \left(\frac{1}{2}\right)^2\\ &= 1 - \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

3. A bag contains 15 marbles: 5 red, 5 blue, and 5 green. A person draws a marble at random, records the color, and then replaces it back in the bag. What is the probability that the person draws a red marble four times in a row?

**Answer:** $\frac{1}{81}$.

4. Nine tiles numbered 1 through 9 are placed in a bag, A tile is randomly drawn and replaced. A second tile is then drawn. What is the probability the first tile drawn and the second tile drawn are both even numbers? **(TB)**

(a) $\frac{1}{9}$

(b) $\frac{2}{9}$

(c) $\underline{\frac{16}{81}}$

(d) $\frac{2}{9}$

**Answer:** $\frac{16}{81}$.

5. Which of the following sets of numbers has the same mean, median, and mode?

   (a) (10, 2, 6, 4, 8)

   (b) (4,3,7,4,2)

   (c) (5,2,3,1,4)

   (d) (8,5,6,7,5)

   **Answer:** (4,3,7,4,2).

6. A special deck of cards contains 20 red cards and 30 black cards. The red cards are split into 10 face cards and 10 number cards, and the black cards into 15 face cards and 15 number cards. A card was selected at random. What is the probability that it was a black face card?

   **Answer:** $\frac{3}{10}$.

## 1.5 Algebra

1. Evaluate the expression $\left(\frac{3a+2b}{2}\right)$ when $a = -3$ and $b = -4$.

   **Solution:** $-8.5$.

$$\frac{3(-3) + 2(-4)}{2} = \frac{-17}{2} = -8.5$$

2. Solve the following simultaneous equations for $x$ and $y$.

$$2x + y = 12$$
$$6x + 5y = 40$$

   **Solution:** $x = 5, y = 2$.

$$3 \times (2x + y = 12)$$
$$-(6x + 5y = 40)$$

$$6x + 3y = 36)$$
$$-(6x + 5y = 40)$$

   This gives $-2y = -4 \rightarrow y = 2$. Substituting this, we get $2x + 2 = 12 \rightarrow 2x = 10 \rightarrow x = 5$.

3. How many real solutions does $3x^2 - 2x + \frac{1}{3} = 0$ have?

(a) 0
(b) cannot be determined
(c) <u>1</u>
(d) 2

**Solution:** 1.

First order of business is to calculate the discriminant, $b^2 - 4ac$.

$$b^2 - 4ac = (-2)^2 - 4(3)(\frac{1}{3}) = 4 - 4 = 0.$$

Luckily for us, this discriminant equals 0, telling us this quadratic equation has exactly one real solution.

4. A car travels from Town A to Town B moving at a constant speed. The car starts at Town A and covers a distance of 1500 metres in 1 min 40 s. The car stops for 15 s and then resumes its journey. If the total journey took 3 min 30 s, what is the distance between the two towns?

   **Solution:** 2925 m.

   Speed of car $= \frac{1500\text{m}}{1\text{min}40s} = \frac{1500\text{m}}{100s} = 15\frac{\text{m}}{s}$.

   Total moving time of journey $= 3$ min $30s - 15s = 3$ min $15s = 195s$

   Total distance between the two towns $= 15\frac{\text{m}}{s} \cdot 195s = 15 \cdot (200 - 5) = 3000 - 75 = 2925\text{m}$

5. For the following quadratic equation determine the **minimum value of y and the x value** for which this occurs: $y = 2x^2 + 12x - 10$

   **Answer:** $x = -3, y = -28$.

6. Given that $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 4x - 1$, what is the value of $f(g(2))$?

   (a) 210
   (b) <u>214</u>
   (c) 212
   (d) 215

   **Solution:** 214.

   $$g(2) = 2^2 + 4(2) - 1 = 4 + 8 - 1 = 11$$

   $$f(11) = 2(11)^2 - 3(11) + 5 = 2(121) - 33 + 5 = 242 - 33 + 5 = 214$$

   $$f(g(2)) = 214$$

7. Three times the sum of two numbers is 222. The square root of their product is 35. What are the two numbers?

   **Answer:** 25, 49.

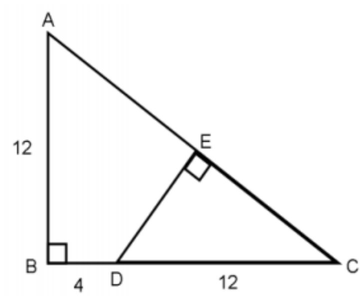
8. $f(x) = x(2 - x)$ is the equation of a parabola. For $x > 1$ the slope of $f(x)$ is:

   (a) 0
   (b) positive
   (c) negative
   (d) a complex number

   **Answer:** negative.

## 1.6 Geometry

1. In the diagram below,




   (a) what is the length of the line $AC$?
   (b) what is the product of the lengths $DE$ and $AC$?

   **Solution:**

   (a) 20 units.

   By Pythagoras' theorem, $AC^2 = AB^2 + BC^2 = 12^2 + (12 + 4)^2 = 144 + 256 = 400$. $AC = \sqrt{400} = 20$.

   (b) 144 units$^2$.

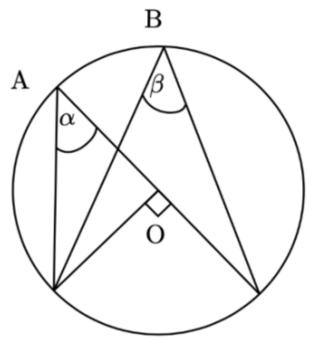
   $\angle ACB = \angle DCE$ and $\angle ABC = \angle CED = 90°$, which gives that $\angle CDE = \angle BAC$. Therefore the triangles $ABC$ and $CDE$ are similar, with side $DE$ corresponding to $AB$ and $DC$ corresponding to $AC$. Therefore, $DE = AB \times \frac{DC}{AC} = 12 \times \frac{12}{20} = \frac{144}{20}$. This gives $DE \times AC = \frac{144}{20} \times 20 = 144$.

2. The length of the diagonal of a square is 1 metre. Compute the area of this square.

   **Solution:** $\frac{1}{2}\text{m}^2$.

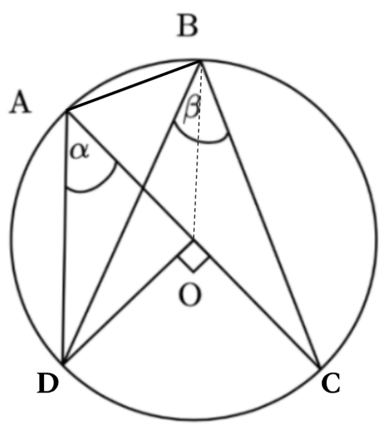
Let the length of a side of the square be $s$. A diagonal drawn forms a right-angled isosceles triangle with two sides $s$. By Pythagoras' theorem, the length of the diagonal (hypotenuse) $= \sqrt{s^2+s^2} = \sqrt{2}s$. If the length of the diagonal is 1, then $\sqrt{2}s = 1 \rightarrow s = \frac{1}{\sqrt{2}}$. The area of a square with sides $s$ is $s^2$, therefore the area of the square is $\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$.

3. You are given the circle below. The angle $\beta$ is bisected by a line $OB$ (not shown). If the diameter of the circle is $D$, what is the length of the arc $AB$?




**Solution:** $\frac{\pi D}{8}$ units.

According to the Central Angle Theorem, the central angle subtended by two points on the circumference of a circle is twice the inscribed angle subtended by those same points. This means for our circle as drawn, the angle $\beta$ equals $\frac{90°}{2} = 45°$.



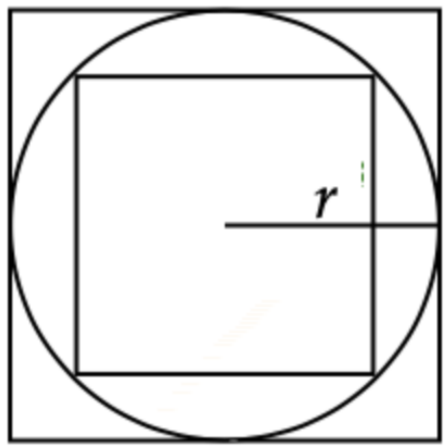

We also use the fact that inscribed angles in a semicircle are 90°. Therefore, $\angle ABC = 90°$.
Since $\beta = 45°$, then $\angle ABD = 90 - 45 = 45°$.
With the bisector $OB$ drawn, we know that $\angle OBA = 45 + 22.5 = 67.5°$.
As both $OB$ and $OA$ are both radii of equal length, let's say, $r$, the $\triangle AOB$ is isosceles with

equal angles $\angle OAB = \angle OBA = 67.5°$.
This gives the $\angle AOB = 180 - 2(67.5) = 180 - 135 = 45°$.

Now that we have this information, we can find the length of the arc $AB$.

$$\text{Arc length} = \pi \cdot D \cdot \frac{45}{360} = \frac{\pi D}{8}.$$

4. A circle of radius $r$ is inscribed in a square. Another square is then inscribed in the circle. What is the area of the space between the 2 squares? **(JH)**




**Solution:** $2r^2$.
Area of large square $=4r^2$

Area of small square$= 2r^2$

Difference between the two squares $= 2r^2$

5. The edges of a rectangular box are 3,4, and 5 inches, respectively. What is the surface area of the box? **(TB)**

**Answer:** 94 square inches.

6. The coordinates of the vertices for rectangle ABCD are A(2,4), B(6,10), C(9,8), D(5,2). What is the length of a diagonal (AC or BD) of the rectangle. **(TB)**

**Answer:** $\sqrt{65}$.

# 2 Level II

## 2.1 Algebra

1. We are given that $x^4 - y^4 = 13$ and $x^2 + y^2 = 4$. What is the value of $(x+y)^2$?

   **Solution:** $4 \pm \frac{1}{4}\sqrt{87}$.

   By the difference of two squares, $(a^2 - b^2) = (a+b)(a-b)$. Therefore, $x^4 - y^4 = (x^2)^2 + (y^2)^2 = (x^2 + y^2)(x^2 - y^2)$. From this, we get that $(x^2 - y^2) = \frac{x^4 - y^4}{x^2 + y^2} = \frac{13}{4}$. We now have

   $$x^2 + y^2 = 4 \tag{3}$$
   $$x^2 - y^2 = \frac{13}{4} \tag{4}$$

   Adding (3) and (4), we get that $2x^2 = \frac{29}{4}$. By (3) - (4), we get that $2y^2 = \frac{3}{4}$. Thus, $4x^2y^2 = \frac{29*3}{4*4} = \frac{87}{16}$. $2xy = \pm\sqrt{\frac{87}{16}}$.

   $$(x+y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2xy = 4 \pm \frac{\sqrt{87}}{4}.$$

2. For what positive real values of $k$ will the equation $kx^2 - (k+1)x - 1 = 0$ have exactly one real solution?

   **Solution:** None.

   First order of business is to calculate the discriminant, $b^2 - 4ac$.

   $$b^2 - 4ac = (-(k+1))^2 - 4(k)(-1) = (k+1)^2 + 4 = 0 \rightarrow (k+1)^2 = -4.$$

   This has no real-valued solutions.

3. Which of the following values of $x$ satisfies the inequality $x^2 + 8x + 15 < 0$?

   (a) $x = -4$
   (b) $x = -2$
   (c) $x = -5$
   (d) $x = -3$

   **Solution:** $x = -4$.

   $$x^2 + 8x + 15 = 0$$
   $$(x+3)(x+5) = 0$$
   $$x + 3 = 0,\ x + 5 = 0$$
   $$x = -3,\ x = -5$$

   In order to satisfy the inequality, values must either be $x < -3$, $x \neq -5$ and $x < -5$. The only provided value which satisfies these conditions is $x = -4$.

4. Find the two values of $x$ for which $x^{1+\log_{10} x} = 10^4 \cdot x$.

**Solution:** $x = 10^2$ and $x = 10^{-2}$.

$$
\begin{aligned}
x^{1+\log_{10} x} &= 10^4 \cdot x \\
x \cdot x^{\log x} &= 10^4 \cdot x \\
x^{\log x} &= 10^4 \\
\text{Let } x &= 10^y. \\
(10^y)^{\log 10^y} &= 10^4 \\
(10^y)^y &= 10^4 \\
10^{y^2} &= 10^4 \\
y^2 &= 4 \\
y &= \pm 2 \rightarrow x = 10^2, 10^{-2}
\end{aligned}
$$

5. Find the smallest positive integer value of $n$ such that $n^2 + 19n + 18$ is divisible by 17.

**Solution:** 16.

$n^2 + 19n + 18 = n^2 + 2n + 1 + 17(n + 1)$. Therefore, we need $(n + 1)^2$ to also be divisible by 17. Since 17 is a prime number, then we need $(n + 1)$ to be divisible by 17 as well. The smallest positive integer value at which this occurs is 16.

## 2.2 Geometry

1. Find the equation of the sphere whose diameter has endpoints $(-2, 3, 1)$ and $(0, 5, -1)$.

**Solution:** $(x + 1)^2 + (y - 4)^2 + z^2 = 3$.

Centre of the sphere = Midpoint of the diameter, $M = \left(\dfrac{-2+0}{2}, \dfrac{3+5}{2}, \dfrac{1-1}{2}\right) = (-1, 4, 0)$.

$$\text{Length of diameter, } l_D = \sqrt{(-2)^2 + (3-5)^2 + (1-(-1)^2} = \sqrt{12} = 2\sqrt{3}.$$
$$\text{Length of radius, } l_r = \sqrt{3}.$$

This gives the equation $(x + 1)^2 + (y - 4)^2 + z^2 = (\sqrt{3})^2 = (x + 1)^2 + (y - 4)^2 + z^2 = 3$.

2. A solid wooden cube is painted red on the outside. The cube is then cut into 27 smaller identical cubes. What fraction of all of the smaller cubes' sides are painted red?

   (a) $\frac{1}{6}$
   (b) $\frac{2}{7}$
   (c) $\underline{\frac{1}{3}}$
   (d) $\frac{1}{2}$

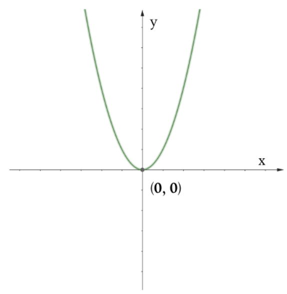
**Solution:** $\frac{1}{3}$.

A cube has 6 faces. In order to cut one larger cube into 27 identical ones, one would need to make 4 cuts (2 top-down and 2 left-right), resulting in the formation of 9 faces for each face of the original cube. This gives $9 \cdot 6 = 54$ painted smaller cube faces. As there are 27 cubes, and a cube has 6 faces, there are a total of 162 faces. The fraction of the total number of sides painted red is $\frac{54}{162} = \frac{1}{3}$.

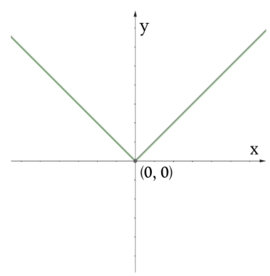
3. Rapid Plot! Sketch the following in 1 minute (note that $a$ is a positive integer): $y = ax^2$, $y = |x|$, $y = e^{-ax}$, $y = \log(\frac{x}{a})$.

**Solution:**

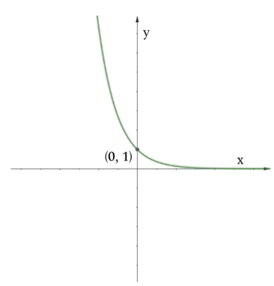
1. $y = ax^2$



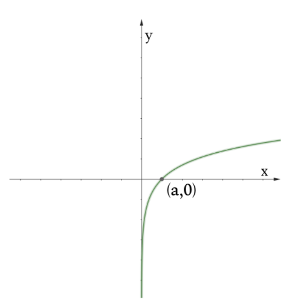

2. $y = |x|$



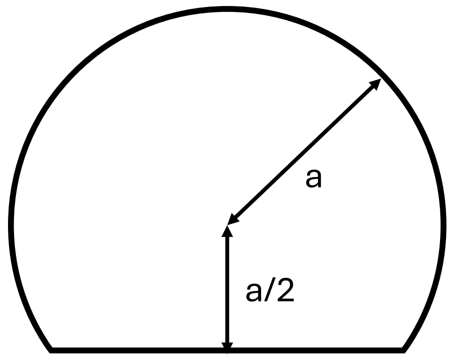

3. $y = e^{-ax}$




4. $y = \log(\frac{x}{a})$

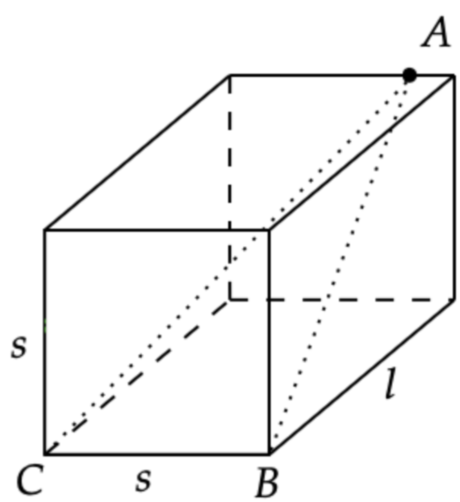


5. A circle of radius $a$ is sawed off at the bottom at a distance of $\frac{a}{2}$ from the edge. What is the area of the remaining portion of the circle.




**Answer:** $\frac{2}{3}\pi a^2 + \frac{\sqrt{3}a^2}{4}$.

6. Consider a square-based cuboid with side lengths $s$ and $l$. What is the area of the triangle constructed using the points $A, B, C$ as seen below




(a) $\frac{s\sqrt{l^2+s^2}}{2}$

(b) $\frac{s\sqrt{l^2-s^2}}{2}$

(c) $\frac{1}{2}s^2$

(d) $\frac{1}{2}sl$

**Answer:** $\frac{s\sqrt{l^2+s^2}}{2}$.

7. Determine the Cartesian equation of a curve defined by the parametric equations $y = 3\sec\theta$ and $x = 3\tan\theta$

   **Answer:** $y^2 - x^2 = 9$.

## 2.3 Trigonometry

1. If $\cos A = \frac{3}{5}$, find $\sin 2A$.

   **Solution:** $\frac{24}{25}$.

$$\sin 2A = \sin(A + A) = 2\sin A\cos A.$$

   Also, recall that $\sin^2 A + \cos^2 A = 1$. Therefore, if $\cos A = \frac{3}{5}$, then $\sin A = \sqrt{1 - (\frac{3}{5})^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Then, $\sin 2A = 2(\frac{3}{5})(\frac{4}{5}) = \frac{24}{25}$.

2. If $\tan(x + y) = \sqrt{3}$ and $\tan(x - y) = \frac{1}{\sqrt{3}}$, where $(x + y) \in (0°, 90°)$ and $x > y$, find $x$ and $y$.

   **Solution:** $x = 45°, y = 15°$.

$$\begin{aligned}\tan(x + y) &= \sqrt{3}\\ \tan(x + y) &= \tan 60°\\ x + y &= 60\end{aligned}$$

$$\begin{aligned}\tan(x - y) &= \frac{1}{\sqrt{3}}\\ \tan(x - y) &= \tan 30°\\ x - y &= 30\end{aligned}$$

   Solving the simultaneous equations, we gather that $x + y + x - y = 90 \to 2x = 90 \to x = 45$. This gives $y = 45 - 30 = 15$.

3. Given that $\sec\theta = \frac{7}{3}$, find the value of $\sin 2\theta$.

   (a) $\frac{12\sqrt{10}}{49}$

   (b) $\frac{4\sqrt{10}}{3}$

   (c) $\frac{3}{7}$

   (d) $\frac{2\sqrt{10}}{7}$

**Solution:** $\frac{12\sqrt{10}}{49}$.

Since $\sec\theta = \frac{1}{\cos\theta}$, $\cos\theta = \frac{3}{7}$. It follows from the definition of cosine that a right-angled triangle with the angle $\theta$ as described has a hypotenuse of 7 units and adjacent side length of 3 units. By Pythagoras' theorem, the length of the opposite side $x$ can be found: $x^2 = 7^2 - 3^2 = 40 \to x = \sqrt{40} = 2\sqrt{10}$ units.

We deduce from this that $\sin\theta = \frac{2\sqrt{10}}{7}$.

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{2\sqrt{10}}{7}\right)\left(\frac{3}{7}\right) = \frac{12\sqrt{10}}{49}.$$

4. Given that $\cos A = \frac{3}{5}$ and $\sin B = \frac{\sqrt{3}}{2}$, where A and B are acute angles, Which of the following is the largest?

   (a) $\frac{1}{2}\sin 2A$
   (b) $\sin A$
   (c) $\cos B$
   (d) $\cos(A + B)$

   **Answer:** $\sin A$.

5. What is the smallest positive value of $x$ that satisfies the equation $2\cos^2(x) - 3\sin(x) = 3$?

   **Answer:** $\frac{7\pi}{6}$ or $210°$.

## 2.4 Vectors

1. Consider a row vector $P$, consisting of 6 elements, and a column vector $Q$, also consisting of 6 elements. In which order can vector multiplication be performed?

   (a) $PQ$ only
   (b) $QP$ only
   (c) Both ways
   (d) Neither

   **Solution:** Both ways.

   $P$ is a $1 \times 6$ vector, and $Q$ is a $6 \times 1$ vector. $PQ$ yields a scalar, while $QP$ yields a square matrix.

2. If $a = 2i + k$ and $b = -3j + 2k$ (where $i$ is the unit vector along the $x$ axis, $j$ is the unit vector along the $y$ axis, and $k$ is the unit vector along the $z$ axis), find the unit vector $u$ that is opposite in direction to $v = 2a - b$.

**Solution:** $u = -\frac{4}{5}i - \frac{3}{5}j + 0k$.

$$a = 2i + k$$

$$b = -3j + 2k$$

$$v = 2a - b$$

$$2a = 2(2i + k) = 4i + 2k$$

$$2a - b = 4i + 2k - (-3j + 2k) = 4i + 3j$$

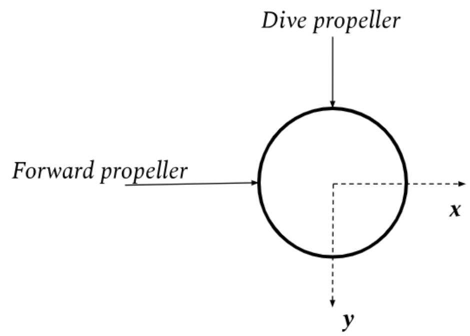
$$v = 4i + 3j, |v| = \sqrt{4^2 + 3^2 + 0^2} = 5$$

By the definition of the unit vector,

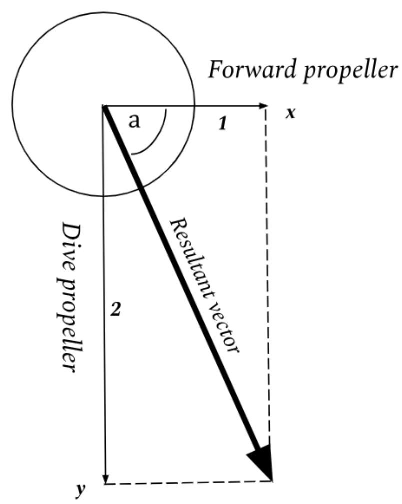
$$u = -\frac{v}{|v|}$$

$$u = -\frac{4}{5}i - \frac{3}{5}j$$

3. A spherical underwater robot has 2 propellers: one to dive and one to move forward. The dive propeller applies a net force (beyond buoyancy) that is twice as strong as the forward propeller. Complete the diagram to show the direction of the robot's movement.




**Solution:**




Resultant vector magnitude: $\sqrt{1^2 + 2^2} = \sqrt{5}$.
Resultant vector angle below horizontal axis: $a = \tan^{-1}(2)$

4. Three 3-dimensional vectors $a, b$ and $c$ satisfy the relationship $a = xb + yc$, Where $x$ and $y$ are scalars. Which of the following is true?

   (a) The vectors are coplanar
   (b) The vectors are collinear
   (c) $a$ is perpendicular to either $b$ or $c$
   (d) None of the above

**Answer:** The vectors are coplanar.

## 2.5 Exponentials and Logarithms

1. The decibel (dB) is a unit of measurement for the loudness of sounds. A $10-$fold increase in loudness corresponds to an increase of $10dB$ ($100-$fold is $+20dB$, $1000-$fold is $+30dB$, and so on).

   (a) Write the relation between the fold-loudness of a sound and its decibel value. You may use $S_0$ and $S_1$ to represent the sound intensities.

   (b) You're sitting at a music mixing board and are told to make the speaker output twice as loud. How many decibels should you increase the output level of the speaker by?

   **Solution:**

   (a) $10 \log_{10} \frac{S_1}{S_0} dB$.

   It's necessary to recognize that the scale is logarithmic. From there, we can write an equation to determine the desired relationship. Let $\frac{S_1}{S_0} = 100$ and calculate given the corresponding decibel increase of $+20$.

$$
\begin{aligned}
20 &= C \log_{10} 100dB \\
C &= 2C \\
C &= 10
\end{aligned}
$$

   Therefore, the relation is $10 \log_{10} \frac{S_1}{S_0}$ dB.

   (b) Increase decibel output by $10 \log_{10} 2$.

$$10 \log_{10} 2x = 10(\log_{10} x + \log_{10} 2) = 10 \log_{10} x + 10 \log_{10} 2.$$

2. If $xy^m = yx^3$, then solve for $m$.

   **Solution:** $m = 1 + 2 \log_x y$.

$$
\begin{aligned}
xy^m &= yx^3 \\
y^{m-1} &= x^2 \\
(m-1) \log y &= 2 \log x \\
m - 1 &= 2 \frac{\log x}{\log y} \\
m &= 1 + 2 \frac{\log x}{\log y}
\end{aligned}
$$

Recognizing this as change of base, we get

$$m = 1 + 2 \log_x y$$

3. Suppose that the population of a colony of bacteria increases exponentially. If the population at the start is 300 and 4 hours later it is 1800, how long will it take for the population to reach 3000?

   (a) $10\frac{\ln 4}{\ln 6}$
   (b) $\underline{4\frac{\ln 10}{\ln 6}}$
   (c) $4\frac{\ln 6}{\ln 10}$
   (d) $6\frac{\ln 10}{\ln 4}$

   **Solution:** $4\frac{\ln 10}{\ln 6}$.

   Let the population of the colony be $P(t)$.

$$P(t) = 300e^{kt}$$
$$1800 = 300e^{4k} \rightarrow 6 = e^{4k} = k = \frac{\ln 6}{4}$$
$$3000 = 300e^{\frac{\ln 6}{4}t} \rightarrow t = \frac{4\ln 10}{\ln 6}$$

4. Find the positive integer solution of $\log_3 x \log_4 x \log_6 x = \log_3 x \log_4 x + \log_4 x \log_6 x + \log_3 x \log_6 x$.

   **Solution:** $x = 72$.

   Divide both sides through by $\log_3 x \log_4 x \log_6 x$. This yields

$$1 = \frac{1}{\log_6 x} + \frac{1}{\log_3 x} + \frac{1}{\log_4 x}$$

   By change of base, we then get

$$\begin{aligned} 1 &= \frac{1}{\frac{\log\, x}{\log\, 6}} + \frac{1}{\frac{\log\, x}{\log\, 3}} + \frac{1}{\frac{\log\, x}{\log\, 4}} \\ &= \frac{\log\, 6}{\log\, x} + \frac{\log\, 3}{\log\, x} + \frac{\log\, 4}{\log\, x} \\ \log\, x &= \log\, 6 + \log\, 3 + \log\, 4 \\ &= \log\, (6 * 3 * 4) = \log\, 72 \end{aligned}$$

   Therefore, $x = 72$.

5. Sketch the function: $y = 1 + x(1 + \frac{x}{2}(1 + \frac{x}{3}(1 + \frac{x}{4}(1 + ...))))$.

   **Solution:** Sketch of $e^x$.

6. In atomic systems at temperature $T$, the ratio $r$, is governed by this equation:

$$r = e^{-\frac{E_1 - E_0}{kT}}$$

   where $k, E_0, E_1$ are constants. Compute the $\Delta T$ needed to double $r$.

(a) $\frac{(E_1-E_0)\ln(2)}{k\ln(r)[\ln(2)+\ln(r)]}$

(b) $\frac{(E_1-E_0)\ln(2)}{k}$

(c) $\frac{(E_1-E_0)\ln(2r)}{k\ln(r)}$

(d) $\frac{-2(E_1-E_0)}{k\ln(2r)}$

**Answer:** $\frac{(E_1-E_0)\ln(2)}{k\ln(r)[\ln(2)+\ln(r)]}$.

7. Find a value for $x$ such that: $\log_3 x + \log_9 x + \log_{27} x = 11$.

   **Answer:** $3^6$.

## 2.6 Statistics and Probability

1. Steve and Diane each have a bag of coloured marbles. Each of their bags contains one red, one blue, one green, one yellow. Steve takes one marble from Diane's bag and places it in his without looking. Diane then takes one marble from Steve's bag and places it into hers without looking. They repeat this process once more. What is the probability that the contents of their bags remain as they were originally?

   **Solution:** $\frac{7}{25}$.

   It does not matter what colour Steve takes at the start of the round since he will regardless have two marbles of the same colour in his bag (but let's say red). After one round, with $\frac{2}{5}$ probability, Diane takes a ball of the colour she is missing (red). For the second round, this probability holds, giving a probability of $\frac{4}{25}$ in this case. Now, let's consider the case that during the first round, Diane takes a different colour than the one she is missing. This happens $\frac{3}{5}$ of the time (let's say it's green this time). Now they each necessarily have two marbles of the same colour in their respective bags. On the second round, Steve has to take one of the two green marbles and Diane has to take one of the two red marbles, which happens with probability $\frac{1}{2} \cdot \frac{2}{5}$. This gives a probability of $\frac{3}{5} \cdot \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{25}$. Putting this all together, the probability the process keeps the contents the same is $\frac{7}{25}$.

2. A bag contains 8 blue balls, 5 red balls and some green balls. If the probability of drawing a green ball is half of the probability of drawing a blue ball, find the probability of drawing a red ball from the bag.

   **Solution:** $\frac{5}{17}$.

   Let $n$ be the number of green balls in the bag. Therefore, the total number of balls in the bag $= n + 8 + 5$.

   So the probability of drawing a green ball is $P_{green} = n/(n+13)$ and the probability of drawing a blue ball is $P_{blue} = 8/(n+13)$

We have $P_{green} = \frac{P_{blue}}{2}$
or
$\frac{n}{n+13} = \frac{8}{2(n+13)}$

Thus $n = 4 =$ number of green balls, and therefore the total number of balls is 17. Thus the probability of drawing a red ball is $\frac{5}{17}$.

3. 20 people attend a meeting. There are 10 men in attendance. At the end of the meeting, everyone shakes hands with everyone else. If we observe a handshake at random, what is the probability that it is a handshake between 2 men?

   **Solution:** $\frac{9}{38}$.

   There are $\binom{20}{2}$ handshakes, $\binom{10}{2}$ of which are between men. Therefore, the probability that a random handshake is between two men is $\frac{\binom{10}{2}}{\binom{20}{2}} = \frac{45}{190} = \frac{9}{38}$.

## 2.7 Calculus

1. A girl sits on a cliff 20 meters above the ground. She throws a tennis ball vertically upwards from the cliff. After $t$ seconds it has height $s$ above the ground which follows:

$$s = 20 + 50t - t^2$$

   What is the maximum height of the ball?

   **Solution:** 645 m.

   To find the maximum height of the ball, we first need to find the value of $t$ for which the value of $s$ is maximized. We do this by differentiating the equation given, equating it to 0, and solving for $t$.

$$\begin{aligned} \frac{ds}{dt} &= 50 - 2t = 0 \\ 2t = 50 &= 25 \\ s(t = 25) &= 20 + 50(25) - (25)^2 \\ s_{max} &= 20 + (25)(25) \\ &= 645 \end{aligned}$$

2. Find the function, $y(x)$, that satisfies the differential equation:

$$\left(\frac{dy}{dx}\right)^2 = \frac{1 - y^2}{1 - x^2}$$

**Solution:** $y = \sin(\arcsin x + C)$.

$$\begin{aligned}
\left(\frac{dy}{dx}\right)^2 &= \frac{1-y^2}{1-x^2} \\
\frac{dy}{dx} &= \sqrt{\frac{1-y^2}{1-x^2}} \\
\sqrt{\frac{1}{1-y^2}}dy &= \sqrt{\frac{1}{1-x^2}}dx \\
\int \sqrt{\frac{1}{1-y^2}}dy &= \int \sqrt{\frac{1}{1-x^2}}dx \\
\arcsin y + C_1 &= \arcsin x + C_2 \\
\arcsin y &= \arcsin x + C \\
y &= \sin(\arcsin x + C)
\end{aligned}$$

3. Find the second derivative of $\sin(x^2 - 4)$ with respect to $x$.

   (a) $4x^2 \sin(x^2-4) - 2\cos(x^2-4)$

   (b) $\underline{-4x^2 \sin(x^2-4) + 2\cos(x^2-4)}$

   (c) $2x^2 \sin(x^2-4) + 2\cos(x^2-4)$

   (d) $2x^2 \sin(x^2-4) + 4\cos(x^2-4)$

   **Solution:** $-4x^2 \sin(x^2-4) + 2\cos(x^2-4)$.

   With chain rule, first derivative: $\frac{d}{dx}(\sin(x^2-4)) = \frac{d}{dx}(x^2-4)\cos(x^2-4) = 2x\cos(x^2-4)$.
   With product and chain rules, second derivative: $\frac{d}{dx}(2x)\cos(x^2-4) + 2x\frac{d}{dx}(\cos(x^2-4))$
   $= 2\cos(x^2-4) + 2x(2x)(-\sin(x^2-4)) = -4x^2\sin(x^2-4) + 2\cos(x^2-4)$.

4. Compute the derivative of $\sqrt{x\ln(x^4)}$.

   **Answer:** $\frac{1+\ln(x)}{\sqrt{x\ln(x)}}$.

5. $f(x) = \frac{\sin 3x}{\cos 2x}$. Find $f'(\frac{\pi}{6})$.

   **Solution:** $4\sqrt{3}$.

$$f(x) = \frac{\sin 3x}{\cos 2x} \to f'(x) = \frac{(\cos 2x)(3\cos 3x) - \sin 3x(-2\sin(2x))}{\cos^2(2x)}$$

   Evaluating at $x = \frac{\pi}{6}$,

$$f'(\frac{\pi}{6}) = \frac{(3\cos 2\frac{\pi}{6})(\cos 3\frac{\pi}{6}) + 2\sin 3\frac{\pi}{6}(\sin(2\frac{\pi}{6}))}{\cos^2(2\frac{\pi}{6})} = \frac{2\sin\frac{\pi}{3}}{\cos^2(\frac{\pi}{3})} = \frac{2\cdot\frac{\sqrt{3}}{2}}{(\frac{1}{2})^2} = 4\sqrt{3}.$$

6. Identify the open intervals in which the function $f(x) = 27x - x^3$ is increasing.

**Solution:** $(-3, 3)$.

$$f'(x) = 27 - 3x^2 = 0$$

$$x = \pm 3$$

$f'(x = -4) = 27 - 3(16) < 0$ indicates decreasing $(-\infty, -3)$
$f'(x = 0) = 27 > 0$ indicates increasing $(-3, 3)$
$f'(x = 4) = 27 - 3(16) < 0$ indicates decreasing $(3, \infty)$

# 3 Level III

## 3.1 Series and Sequences

1. What is the sum of the infinite series

$$\frac{7}{4} - \frac{7}{4^2} + \frac{7}{4^3} - \cdots ?$$

**Solution:** $\frac{7}{5}$.

The general form of an infinite geometric series is $a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots$.

$$r = \frac{-\frac{7}{4^2}}{\frac{7}{4}} = \frac{\frac{7}{4^3}}{-\frac{7}{4^2}} = -\frac{1}{4}.$$

Since $|r| < 1$, the sum of this series is $S = \frac{a_1}{1-r}$.

$$S = \frac{\frac{7}{4}}{1 - \left(-\frac{1}{4}\right)} = \frac{\frac{7}{4}}{\frac{5}{4}} = \frac{7}{5}.$$

2. The sequence $x_n = \frac{1}{n}, n \in \mathbb{Z}^+$ is:

   (a) Bounded and Convergent
   (b) Bounded and Divergent
   (c) Unbounded and Convergent
   (d) Unbounded and Divergent

**Solution:** Bounded and Convergent.

- A sequence is bounded if there exists two numbers such that all the terms of the sequence lie within the range produced by these two numbers. In other words, the sequence must be bounded above and below.

  The largest value in this sequence is 1, and there is no value in this sequence that is less than 0. Therefore, any number greater than 1 serves as an upper bound for this sequence. Any number less than 0 serves as a lower bound for this sequence. Since this is bounded above and below, the sequence is **bounded**.

- A sequence is convergent if the value of the terms of the sequence tend to a particular value.

  As $n \to \infty$, $\frac{1}{n} \to 0$. Therefore, this sequence is **convergent**.

3. Write $\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$ as a complex number in $a + bi$ form. **(VV)**

**Solution:** $\frac{4}{5} + \frac{2}{5}i$.

The sum of an infinite geometric series with $|r| < 1$ :

$$\frac{1}{1-r} = \frac{1}{1-\frac{i}{2}} = \frac{1+\frac{i}{2}}{(1-\frac{i}{2})(1+\frac{i}{2})} = \frac{1+\frac{i}{2}}{\frac{5}{4}} = \frac{4}{5} + \frac{2}{5}i.$$

4. Let $a_i$ be real numbers. Define a sequence using the following recurrence relation: $a_{i+1} = \frac{5a_i}{a_{i-1}}$, and let $a_1 = 2$ and $a_2 = 3$. Find $a_{2024}$.

**Solution:** 3.

$$\begin{aligned}
a_1 &= 2, a_2 = 3 \\
a_3 &= \frac{5(3)}{2} = \frac{15}{2} \\
a_4 &= \frac{5(\frac{15}{2})}{3} = \frac{25}{2} \\
a_5 &= \frac{5(\frac{25}{2})}{\frac{15}{2}} = \frac{25}{3} \\
a_6 &= \frac{5(\frac{25}{3})}{\frac{25}{2}} = \frac{10}{3} \\
a_7 &= \frac{5(\frac{10}{3})}{\frac{25}{3}} = 2 \\
a_8 &= \frac{5(2)}{\frac{10}{3}} = 3
\end{aligned}$$

It then becomes clear that this relation yields a cyclic sequence with a period of 6. Therefore

$$a_{2024} = a_{2024 \bmod 6}.$$

2022 is the closest multiple of 6 to 2024. Deduce this by noting that 2022's digits sum to 6 $(2 + 0 + 2 + 2 = 6)$, which is divisible by 3, thus showing that 2022 itself is divisible by 3, according to a popular divisibility rule.

*This rule states that if the sum of the digits of a number is a multiple of 3, the number is also divisible by 3.* 2022 is an even number, divisible by 2. Since 2 and 3 are both prime numbers, then a number divisible by both is also divisible by their product, yielding the fact that 2022 is divisible by 6.

This can also be determined by dividing 2024 by 6 and observing that you get a remainder of 2. Thus, $a_{2024} = a_2 = 3$.

5. What is $\sum_{n=1}^{\infty} nx^{n-1}$ in its closed form, where $n \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, $|x| < 1$.

**Solution:** $\frac{1}{(1-x)^2}$.

The key insight is:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right)$$

The two operations, summation and differentiation, are independent of each other. The sum on the right-hand side is a convergent geometric series since $|x| < 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty} nx^{n-1} &= \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) \\
&= \frac{d}{dx}\left(\frac{x}{1-x}\right) \\
&= \frac{(1-x)(1) - x(-1)}{(1-x)^2} \\
\sum_{n=1}^{\infty} nx^{n-1} &= \frac{1}{(1-x)^2}
\end{aligned}$$

## 3.2 Euclidean Geometry

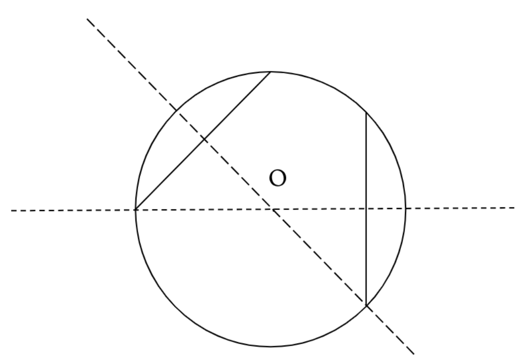
1. Consider a circle with three distinct points labelled $A, B$ and $C$ on its circumference. What is the minimum number of straight lines/line segments which must be drawn to locate the centre of this circle?

   (a) 6 lines
   (b) 2 lines
   (c) 3 lines
   (d) 4 lines

   **Solution:** 4 lines.

   The correct answer is 4 lines. You can find the center by following this procedure:

   - Start by drawing two chords on the circle (shown as solid lines here).
   - Then draw perpendicular bisectors of each of the original chords.
   - Where the two bisectors intersect is the center of the circle.



2. Let $A$ be a point in $\mathbb{R}^3$ with Cartesian coordinates $(x_0, y_0, z_0)$. $A$ can be expressed in spherical coordinates as $(\rho_0, \theta_0, \phi_0)$. The following equations can be used to convert from spherical coordinates to Cartesian coordinates:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases} \qquad \text{for } \rho \geq 0, \theta \in [0, 2\pi) \text{ and } \phi \in [0, \pi).$$

Find $\phi_0(x_0, y_0, z_0)$.

**Solution:** $\phi_0 = \arctan\left(\frac{\sqrt{x_0^2+y_0^2}}{z_0}\right)$.

$$\begin{aligned} x_0^2 + y_0^2 &= \rho_0^2 \sin^2\phi_0 \cos^2\theta_0 + \rho_0^2 \sin^2\phi_0 \sin^2\theta_0 \\ &= \rho_0^2 \sin^2\phi_0 (\cos^2\theta_0 + \sin^2\theta_0) \\ &= \rho_0^2 \sin^2\phi_0 \\ \sqrt{x_0^2 + y_0^2} &= \rho_0 \sin\phi_0 \end{aligned}$$

$$z_0 = \rho_0 \cos\phi_0$$

Combining, we get

$$\begin{aligned} \frac{\sqrt{x_0^2 + y_0^2}}{z_0} &= \frac{\rho_0 \sin\phi_0}{\rho_0 \cos\phi_0} \\ &= \tan\phi_0 \\ \tan\phi_0 &= \frac{\sqrt{x_0^2 + y_0^2}}{z_0} \\ \phi_0 &= \arctan\left(\frac{\sqrt{x_0^2 + y_0^2}}{z_0}\right). \end{aligned}$$

3. Find the length of the major axis of the ellipse which is parametrically defined as:

$$\begin{cases} x = 2 + 3\cos\theta \\ y = 3 + 2\sin\theta \quad \theta \in [0, 2\pi) \end{cases}$$

**Solution:** 6 units.

These parametric equations can be combined to yield the cartesian equation for the ellipse:

$$\begin{aligned} \frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{2^2} &= \cos^2\theta + \sin^2\theta \\ \frac{(x-2)^2}{3^2} + \frac{(y-3)^2}{2^2} &= 1 \end{aligned}$$

By observation, the length of the semi-major axis is 3 units. Therefore, the length of the major axis is twice this length: 6 units.

4. **(VV)**

   (a) What shape is the intersection of the surface $x^2+y^2+z^2 = 1$ with the surface $x+y+z = 1$?

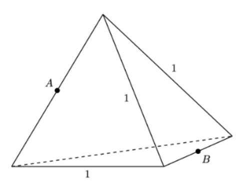
   (b) What is the area of this shape?

   **Solution:**

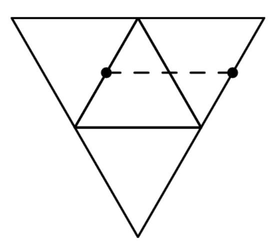
   (a) Circle.
   If a sphere and a plane intersect, the points of intersection must form a circle by symmetry.

   (b) $\frac{2}{3}\pi$.
   Look at the plane from the side and view it's normal vector. From this, we can note that the shortest distance from the origin to the the plane is from $(0, 0, 0)$ to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, a distance $d = \frac{1}{\sqrt{3}}$. By Pythagorean theorem, the radius of the circle follows: $d^2 + r^2 = R^2 = 1 \rightarrow r^2 = 1 - \frac{1}{3} = \frac{2}{3}$. Area follows as $\pi r^2 = \frac{2}{3}\pi$

5. An ant is walking on the surface of a regular solid tetrahedron with all side lengths equal to 1. It wants to get from point $A$ to point $B$ by walking as little as possible. What is the shortest possible distance the ant can walk? **(VV)**




**Solution:** 1.



If you can imagine unfolding the tetrahedron as shown (all the triangles shown remain faces of the tetrahedron), the line joining the points $A$ and $B$ is the same length as a side of the tetrahedron, therefore making the shortest distance between the points along the surface 1.

### 3.3 Statistics and Probability

1. What is the expected value of a game in which a coin is tossed 100 times where you gain \$1.15 every time a head is flipped and lose \$0.55 for every tail? **(HB)**

   **Solution:** \$30.

   $$\frac{1}{2}(100)\$1.15 + \frac{1}{2}(100)(-\$0.55) = \$30$$

2. What is the probability that a random distinguishable arrangement of the word CARIBBEAN has no two A's appearing next to each other? **(CR)**

   **Solution:** $\frac{7}{9}$.

   There are two A's and 7 other letters. To avoid placing the A's next to each other, we can think of placing each of them into slots at the beginning and end of the word, or between 2 other letters. Thus there are 8 possible positions the A's could be placed in, giving us $\binom{8}{2} = 28$ allowed positions for the A's. You can then arrange the remaining letters in $\frac{7!}{2!} = 2520$ ways. We divide by two factorial because we overcount by the number of ways the two B's can be arranged in any given word. This gives $28 \times 2520$ ways we can arrange the letters of CARIBBEAN without having consecutive A's. There are similarly $\frac{9!}{2!2!}$ ways to arrange these letters without any type of restriction. Therefore, the probability that a random arrangement does not have consecutive A's is

   $$\frac{\binom{8}{2} * \frac{7!}{2!}}{\frac{9!}{2!2!}} = \frac{\frac{8\cdot 7}{2!} * \frac{7!}{2!}}{\frac{9!}{2!2!}} = \frac{\frac{8!\cdot 7}{2!2!}}{\frac{9!}{2!2!}} = \frac{7}{9}$$

3. The average number of students who show up to the organic chemistry lecture on Fridays is 4. What is the probability that next Friday at least 2 students show up? State the probability as an expression in $e$.

   (a) $\underline{1 - 5e^{-4}}$
   (b) $2e^{-4}$
   (c) 0.5
   (d) $1 - 4e^{-2}$

   **Solution:** $1 - 5e^{-4}$.

   Let the number of students who show up to the lecture be represented as a Poisson-distributed random variable $X$.

   The Poisson distribution models the number of events occurring in a fixed interval of time. This distribution has parameter $\lambda$, representing the average rate at which the event in question occurs. For our problem, $\lambda$ equals the average number of students who show up to lecture on Fridays, which is 4.

The probability mass function of the Poisson, defining the probability of observing exactly $k$ events is

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Now, to find the desired probability.

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X < 2)$$

$$\mathbb{P}(X \geq 2) = 1 - (\mathbb{P}(X = 1) + \mathbb{P}(X = 0))$$

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 1) - \mathbb{P}(X = 0) = 1 - 4e^{-4} - e^{-4} = 1 - 5e^{-4}.$$

### 3.4 Calculus

1. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin\theta} d\theta$.

   **Solution:** $\ln 2$.

   Let $u = 1 + \sin\theta \rightarrow du = \cos\theta d\theta$. When $\theta = \frac{\pi}{2}, u = 1 + \sin\frac{\pi}{2} = 2$ and $\theta = 0, u = 1 + \sin 0 = 1$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin\theta} d\theta &= \int_1^2 \frac{du}{u} \\ &= \ln|u|\Big|_1^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2.\end{aligned}$$

2. Integrate $\int x\cos x dx$.

   (a) $x\cos x - \sin x$
   (b) $x\sin x - \cos x$
   (c) $\underline{x\sin x + \cos x}$
   (d) $x\cos x + \sin x$

   **Solution:** $x\sin x + \cos x$.

   By the product rule of integration,

$$\begin{aligned}\int x\cos x dx &= x\int \cos x dx - \int\left[\frac{d}{dx}(x)\int \cos x dx\right]dx \\ &= x\sin x + C_1 - \int \sin x dx \\ &= x\sin x + C_1 + \cos x + C_2 \\ &= x\sin x + \cos x + C\end{aligned}$$

3. Simplify $\int \sin^3(t)\cos^2(t)dt$.

**Solution:** $-\frac{\cos^3(t)}{3} + \frac{\cos^5(t)}{5} + C$.

$$\int \sin^3(t)\cos^2(t)dt = \int \sin^2(t)\cos^2(t)\sin(t)dt$$

Recall that $1 - \cos^2(t) = \sin^2(t)$. Substituting this, we get

$$\int \sin^3(t)\cos^2(t)dt = \int (1-\cos^2(t))\cos^2(t)\sin(t)dt$$

Let $u = \cos t$. Then, $du = -\sin t dt \rightarrow \sin t dt = -du$.

$$\begin{aligned}\int \sin^3(t)\cos^2(t)dt &= -\int (1-u^2)u^2 du \\ &= -\int u^2 - u^4 du \\ &= -\Big[\frac{u^3}{3} - \frac{u^5}{5} + C\Big] \\ &= -\frac{\cos^3(t)}{3} + \frac{\cos^5(t)}{5} + C.\end{aligned}$$

4. Solve the integral. **(AU)**

$$\int \frac{e^x}{e^{2x} + 3e^x + 2}dx$$

**Solution:** $-\ln(e^x + 2) + \ln(e^x + 1) + C$.

Let u = $e^x$; du = $e^x dx$.

$$= \int \frac{1}{u^2 + 3u + 2}du$$

$$= \int \frac{1}{(u+2)(u+1)}$$

$$\frac{1}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1} \rightarrow A = -1, B = 1$$

$$\int (-\frac{1}{u+2} + \frac{1}{u+1})du = -\ln|u+2| + \ln|u+1|$$

$$\int \frac{e^x}{e^{2x} + 3e^x + 2}dx = -\ln(e^x + 2) + \ln(e^x + 1) + C$$

5. If $x = \tanh y$, derive the equation for $\tanh^{-1} x$ in terms of $x$.

**Solution:** $\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1.$

$$\begin{aligned} x &= \frac{\sinh y}{\cosh y} = \frac{\frac{1}{2}(e^y - e^{-y})}{\frac{1}{2}(e^y + e^y)} \\ &= \frac{e^{2y}-1}{e^{2y}+1} \end{aligned}$$

Thus $xe^{2y} + x = e^{2y} - 1$. Therefore, $e^{2y} = \frac{1+x}{1-x}$

$$\begin{aligned} \ln e^{2y} &= \ln\frac{1+x}{1-x} \\ y &= \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1 \end{aligned}$$

$\tanh^{-1} x = \tanh^{-1}(\tanh y) \to y = \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad |x| < 1.$

## 3.5 Vectors and Matrices

1. Find the determinant of the matrix $\begin{bmatrix} 1 & -4 & 7 & 5 \\ 0 & 3 & 1 & -6 \\ 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

   **Solution:** $-6$.

$$\begin{vmatrix} 1 & -4 & 7 & 5 \\ 0 & 3 & 1 & -6 \\ 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 & -6 \\ -1 & 7 & 0 \\ 0 & 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 & -6 \\ 1 & 7 & 0 \\ 0 & 0 & 2 \end{vmatrix} + 7\begin{vmatrix} 0 & 3 & -6 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 & 1 \\ 1 & -1 & 7 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\begin{aligned} \begin{vmatrix} 1 & -4 & 7 & 5 \\ 0 & 3 & 1 & -6 \\ 1 & -1 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} &= 1\left(3\begin{vmatrix} 7 & 0 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} -1 & 7 \\ 0 & 0 \end{vmatrix}\right) \\ &+ 4\left(0\begin{vmatrix} 7 & 0 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}\right) \\ &+ 7\left(0\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} - 6\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\right) \\ &- 5\left(0\begin{vmatrix} -1 & 7 \\ 0 & 0 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}\right) \\ &= (3(14) - 1(-2) + 0) + 4(0 - 1(2) + 0) + 7(0 - 3(2) + 0) - 5(0 + 0 + 0) \\ &= 42 + 2 - 8 - 42 = -6. \end{aligned}$$

2. The eigenvalues, $\lambda$, of an $n \times n$ matrix, $A$, can be found by solving the equation $det(A - \lambda I) = 0$ where $I$ is the $n \times n$ identity matrix. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$.

**Solution:** $\lambda_+ = \frac{5+i\sqrt{3}}{2}$ and $\lambda_- = \frac{5-i\sqrt{3}}{2}$.

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (2-\lambda)(3-\lambda) - 1(-1) = 0$$

$$6 - 5\lambda + 1 + \lambda^2 = 0 \rightarrow \lambda^2 - 5\lambda + 7 = 0$$

Solving this quadratic equation, we get that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{5^2 - 4(7)}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5 \pm i\sqrt{3}}{2}.$$

3. Calculate $(\vec{a} \times \vec{b}) \cdot \vec{c}$ where $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{c} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. **(VV)**

   **Solution:** 0.

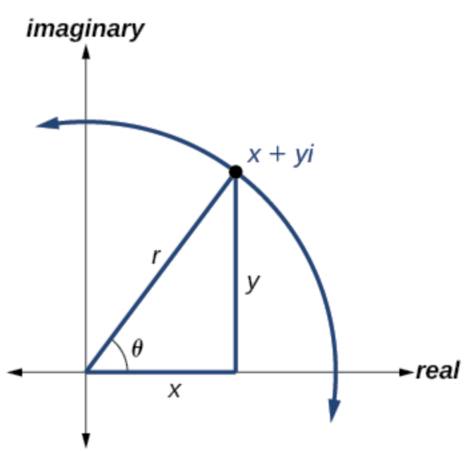
   Notice that $\vec{c} = \vec{a} - \vec{b}$ so $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{a} \times \vec{b}) \cdot (\vec{a} - \vec{b}) = (\vec{a} \times \vec{b}) \cdot \vec{a} - (\vec{a} \times \vec{b}) \cdot \vec{b} = 0$ as $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

## 3.6 Complex Numbers

1. The complex number, $z$, can be represented by a vector in the complex plane. What is the effect on the vector representing $z$ in the complex plane, of multiplying $z$ by $re^{i\theta}, r \in (0, \infty), \theta \in [-\pi, \pi)$?

   **Solution:** Multiplying $z$ by $re^{i\theta}$ will make the vector $r$ times as long and rotate it an angle of $\theta$ counterclockwise around the origin.

   Consider a complex number in polar form: $z = r_1 e^{i\theta_1}$. This means that the complex number when represented on the complex plane is a line with magnitude (length $r$) positioned at an angle of $\theta$ above the positive real axis as shown.



With this knowledge, we can see that if we multiple the complex number $z$ by $re^{i\theta}$, we get

$$r_1 e^{i\theta_1} \cdot re^{i\theta} = r_1 r e^{i(\theta+\theta_1)}$$

which represents a new complex number of magnitude $rr_1$ and argument $\theta + \theta_1$ above the positive real axis, as our answer describes.

2. Find the purely real solution to $4z^2 + (8+4i)z + (3+2i) = 0$.

   **Solution:** $z = -\frac{1}{2}$.

   This equation can be solved directly using the quadratic formula of course, but this is a rather tedious approach. You can more quickly solve this by recognizing that only the real solution of the equation is desired. Therefore, let $z = a$, where $a \in \mathbb{R}$. The equation becomes:

$$4a^2 + (8+4i)a + (3+2i) = 0$$

$$4a^2 + 8a + 3 + i(4a+2) = 0$$

   For this equation to be satisfied, both the real and imaginary part of the complex number must equal 0. For this to be the case, $4a + 2 = 0$ and $4a^2 + 8a + 3 = 0$. The only value that satisfies both equations is $z = -\frac{1}{2}$.

## 3.7 Differential Equations

1. Find the general solution for the differential equation $\frac{dy}{dx} = x + xy^2$.

   **Solution:** $y = \tan\left(\frac{x^2}{2} + C\right)$.

   We proceed by separation of valuables.

$$\begin{aligned}
\frac{dy}{dx} &= x + xy^2 \\
dy &= x(1+y^2)dx \\
\frac{1}{1+y^2}dy &= xdx \\
\int \frac{1}{1+y^2}dy &= \int xdx \\
\arctan y &= \frac{x^2}{2} + C \\
y &= \tan\left(\frac{x^2}{2} + C\right)
\end{aligned}$$

2. Solve the differential equation $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, where $x(0) = 0$ and $\frac{dx}{dt}(0) = 5$.

   (a) $5e^{-t} - 5e^{6t}$

   (b) $5e^{-t} + 5e^{6t}$

(c) $5e^{-2t} + 5e^{-3t}$

(d) $\underline{5e^{-2t} - 5e^{-3t}}$

**Solution:** $5e^{-2t} - 5e^{-3t}$.

Let $x_h(t)$ be of the form $x_h(t) = Ae^{\lambda t}$ where both $A$ and $\lambda$ can be complex. Then,

$$\lambda^2 A + 5A\lambda + 6A = 0$$

Solving for $\lambda$, we get

$$\begin{aligned} \lambda^2 A + 5A\lambda + 6A &= 0 \\ (\lambda_1 + 2)(\lambda_2 + 3)A &= 0 \\ \lambda_1 &= -2, \lambda_2 = -3. \end{aligned}$$

Hence, $x(t) = A[e^{-2t} + e^{-3t}] = C_1 e^{-2t} + C_2 e^{-3t}$.

When $x(0) = 0$, $0 = C_1 + C_2 = 0 \rightarrow C_1 = -C_2$.

We also know that $\frac{dx}{dt}(0) = 5$, therefore

$$\begin{aligned} -2C_1 e^{-2(0)} - 3C_2 e^{-3(0)} &= 5 \\ -2C_1 - 3C_2 &= 5 \\ -2C_1 + 3C_1 &= 5 \\ C_1 &= 5 \\ C_2 = -C_1 &= -5 \end{aligned}$$

Therefore, solution is $5e^{-2t} - 5e^{-3t}$.

3. Find the general solution for the following underdamped system: $x'' + 4x' + 8x = 0$.

**Solution:** $x_h = 2Ae^{-2t} \cos(2t + \phi)$. **Alternate solution:** $x_h = e^{-2t}[C_1 \sin(2t) + C_2 \cos(2t)]$.

(a) Let $x_h(t)$ be of the form $x_h(t) = Ae^{\lambda t}$ where both $A$ and $\lambda$ can be complex. Then,

$$\lambda^2 A + 4A\lambda + 8A = 0$$

Solving for $\lambda$,

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)} = -2 \pm \frac{\sqrt{-16}}{2} = -2 \pm 2i.$$

i.e, $\lambda_1 = -2 + 2i$ and $\lambda_2 = -2 - 2i$.

This gives $x_h(t) = Ae^{\lambda t} = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$. Hence, $x(t) = Ae^{j\phi} e^{-2t}[e^{i2t} + e^{-i2t}]$. The homogenous solution is therefore $x_h(t) = 2Ae^{-2t} \cos(2t + \phi)$.

(b) Alternatively, consider $x(t) = Ae^{j\phi}e^{-2t}[e^{i2t} + e^{-i2t}]$. By Euler's formula,

$$e^{ix} = \cos x + i \sin x.$$

Therefore,

$$\begin{aligned} x(t) &= Ae^{j\phi}e^{-2t}(e^{i2t} + e^{-i2t}) \\ &= A_1e^{-2t}e^{i2t} + A_2e^{-2t}e^{-i2t} \\ &= A_1e^{-2t}(\cos 2t + i \sin 2t) + A_2e^{-2t}(\cos 2t - i \sin 2t) \\ &= e^{-2t}\Big((A_1 + A_2)\cos 2t + i(A_1 - A_2)\sin 2t\Big). \end{aligned}$$

This solution can also be expressed as $x_h(t) = e^{-2t}\Big(C_1 \sin(2t) + C_2 \cos(2t)\Big)$, where $C_1 = i(A_1 - A_2)$ and $C_2 = A_1 + A_2$.

# 4 Note

Keep in mind that there are many ways to solve a mathematics problem. These are just examples which will lead to the right solution. Try to think of other ways to approach the problems!